SINGULARITIES OF THE PROJECTIONS OF n-DIMENSIONAL KNOTS

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Abstract. Let $n \geqq 5$. There is a smoothly knotted n-dimensional sphere in $(n+2)$-space such that the singular point set of its projection in $(n+1)$-space consists of double points and that the components of the singular point set are two. (The sphere is *knotted* in the sense that it does not bound any embedded $(n+1)$-ball in $(n+2)$-space.) Furthermore, the projection is not the projection of any unknotted sphere in $(n+2)$-space. There are two inequivalent embeddings of an n-manifold in $(n+2)$-space such that the projection of one of these in $(n+1)$-space has no double points and the projection of the other has a connected embedded double point set.

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1.Introduction and Main results

In the study of classical knots, the projections of a knot into $\mathbb{R}^2$ plays an important role (see e.g. [A], [BL], [C], [CF], [J], [Kf1], [Ko], [Re], [V], [W], for example). For 2-dimensional knots in $\mathbb{R}^4$, the projection in $\mathbb{R}^3$ is considered (see [CS2], [Km], [KSS], [Su], for example.). Projections of codimension two submanifolds of $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$ is studied by [CS1], [CS3], [G], [R], etc. Projections of p-dimensional submanifolds of $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$, for $p < n-2$, is studied by [SS], etc.

In this paper, we consider the projection (into $\mathbb{R}^{n+1}$) of n-dimensional embeddings in $\mathbb{R}^{n+2}$ where $n \geqq 5$. We work in the smooth category throughout. We consider those embeddings for which the projection has relatively simple self-intersections. We show that there are embeddings that are truly knotted, but whose projections have simple self intersections. First we introduce some notation.

We work in the smooth category.

An *(oriented) n-(dimensional) knot* K is a smooth oriented submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}$ which is diffeomorphic to the standard n-sphere. We say that n-knots K_1 and K_2 are *equivalent* if there exists an orientation preserving diffeomorphism $f : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1} \times \mathbb{R}$ such that $f(K_1)=K_2$ and $f|_{K_1} : K_1 \to K_2$ is an orientation preserving diffeomorphism.

Note. In many other papers, including the author's, the definition of n-knot is a smooth oriented submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}$ which is PL homeomorphic to the standard n-sphere. But in this paper, we adopt the former one and reject the latter one.

Let T be the unit n-sphere of $\mathbb{R}^{n+1} \times \{0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$. Then T is an n-knot. An n-knot K is said to be *unknotted* if K is equivalent to T. If K is not unknotted, then we say that K is *truly knotted*.

Let π: $\mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1} \times \{0\}$ be the natural projection map. We suppose $\pi|_K$ is a self-transverse immersion. The *projection* P of an n-knot K is $\pi|_K(K)$ of $\mathbb{R}^{n+1}$. We give P an orientation by using the orientation of K naturally. The *singular point set* of the projection of an n-knot K is the set $\{x \in \pi|_K\ (K) \mid \sharp\ \{(\pi|_K)\ ^{-1}(x)\} \geqq 2.\}$. Let $\mu(P)$ denote the number of the connected components of the singular point set of the projection P.

Let K be an n-knot with a projection P. Then the number $\mu(P)$ measures the complexity of K as follows.

Let n=1. If $\mu(P) \leqq 2$, then K is unknotted. (It is proved by chcking all posibble projections concretely.)

Let n=2. Suppose the singular point set of P consists of double points. If $\mu(P) \leqq 2$, then K is unknotted.

Let n be any natural number. There is an n-knot K with a projection P with the following properties. (1)$\mu(K)$=3 (2)K is truly knotted. (3)The singular point set of P consists of double points. Proof. Let K_1 be the trefoil knot. Let K_n be the (0-twist) spun knot of K_{n-1} ($n \geqq 2$). (See [Z] for twist spun knots.)

It is natural to consider the following problem.

Problem A. Let K be an n-knot with a Projection P (thus the underlying manifold K is an n-sphere). Suppose the singular point set of P consists of double points. Suppose that $\mu(P) \leqq 2$. Then, is K unknotted?

Of course, if $n = 1$ or 2, as mentioned above, then the answer is affirmative. But for general $n \geq 5$, we prove the answer to Problem A is negative in §3.

We prove:

Theorem 1. *Let $n \geqq 5$. There is an n-knot K with a projection P with the following properties.*

(1) K is diffeomorphic to the standard n-sphere.

(2) The singular point set of P consists of double points.

(3) $\mu(P) = 2$.

(4) K is truly knotted.

In §4, furthermore, we prove the projection of the n-knot constructed in the proof of Theorem 1 has the following property.

Theorem 2. *Let $n \geqq 5$. There is an n-knot with a projection P such that P is not the projection of any knot which is unknotted.*

Note. (1) It is well-known that the projection of any 1-dimensional knot is the projection of a 1-knot which is unknotted. The fact is used in definitions of the Jones polynomial and the Conway-Alexander polynomial. See [Kf1] and [Kf2].

(2) The author proved the $n \geqq 3$ case of Theorem 2 is true in [O]. But $\mu(P)$ of the examples are greater than two.

In the case of codimension two submanifolds of $\mathbb{R}^{n+2}$ which are diffeomorphic to a connected closed manifold and which are not spheres, we have the following Problem B corresponding to Problem A.

Let M be a connected closed n-manifold. Let K be a submanifold of $\mathbb{R}^{n+2}$ which is diffeomorphic to M. Suppose $\pi|K$ is transverse. Put $P = \pi(K)$. The number $\mu(P)$ is defined similarly.

Submanifolds K_1 and K_2 in $\mathbb{R}^{n+2}$ are said to be *equivalent* if there is a diffeomorphism $f : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ such that $f(K_1) = K_2$ and that $f|K_1$ is an orientation preserving diffeomorphism if M is oriented.

Problem B. Let M, K, P and $\mu(\quad)$ be as above. Suppose the singular point set of P consists of double points. Suppose that $\mu(P) \leqq 2$ (resp. $\leqq 1$). Then, is an equivalence class of submanifolds determined uniquely? In particular, is it determined uniquely when M is embedded in $\mathbb{R}^{n+1}$?

[Sh] shows that, when $M \cong T^2$, then the equivalence class of submanifolds is determined provided $\mu(\quad) \leqq 2$. On the other hand, for high dimensional case we have the following.

In §2 we prove:

Theorem 3. *Let $n \geqq 5$. There is a closed connected oriented n-dimensional manifold M as follows. There are submanifolds K_i with a projection P_i $(i = 0, 1)$ which are diffeomorphic to M with the following properties.*

(1) $\mu(P_0)$=0.

(2) $\mu(P_1)$=1.

(3) The singular point set of P_i consists of double points.

(4) K_0 is equivalent to neither K_1, $-K_1$, K_1^* nor $-K_1^*$.

(5) M is embedded in $\mathbb{R}^{n+1}$.

The construction of the manifold M in Theorem 3 will be used in the proofs of Theorem 1 and 2.

2. The proof of Theorem 3

We first prove the case of $n = 5$.

We define submanifolds K_0 and $K_1 \subset \mathbb{R}^7{=}\mathbb{R}^6 \times \mathbb{R}^1$ which are diffeomorphic to $S^3 \times S^2$. Of course $S^3 \times S^2$ is embedded in $\mathbb{R}^6$.

We define $K_0 \subset \mathbb{R}^7{=}\mathbb{R}^6 \times \mathbb{R}^1$. Let A_0 be a trivially embedded 3-sphere in $\mathbb{R}^6 \times \{0\}$. Take the tubular neighborhood N_0 of A_0 in $\mathbb{R}^6 \times \{0\}$. Then ∂N_0 is diffeomorphic to $S^3 \times S^2$. Define K_0 to be ∂N_0. The projection P_0 of K_0 coincides with K_0. Obviously $\mu(P_0){=}0$.

We define $K_1 \subset \mathbb{R}^7 = \mathbb{R}^6 \times \mathbb{R}^1$. Take a self-transverse immersion $g : S^3 \looparrowright \mathbb{R}^6 \times \{0\}$ such that the singular point set is one point p. Then $\sharp\{g^{-1}(p)\} = 2$. We suppose that there is a subset V of $\mathbb{R}^6 \times \mathbb{R}^1$ with the following properties.

(1) $V = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 < 1,\ y_1^2 + y_2^2 + y_3^2 < 1,\ z \in \mathbb{R}.\}$.

(2) $V \cap g(S^3)$ is a union of two open 3-discs D_x^3 and D_y^3.

(3) $D_x^3 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 < 1,\ y_1{=}y_2{=}y_3{=}0,\ z = 0\ \}$.

(4) $D_y^3 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1{=}x_2{=}x_3{=}0,\ y_1^2 + y_2^2 + y_3^2 < 1,\ z = 0\ \}$.

Take the normal bundle ν of $g(S^3)$ in $\mathbb{R}^6 \times \{0\}$. Let E be a manifold which is the total space of ν. Thus we obtain an immersion $\widetilde{g} : E \looparrowright \mathbb{R}^6 \times \{0\}$. Since $\pi_2 SO(3){=}0$, ν is the trivial bundle and ∂E is diffeomorphic to $S^3 \times S^2$.

We can take $\widetilde{g}$ to satisfy the following conditions.

(1) $\widetilde{g}|_{V^C}$ is an embedding, where V^C is $\widetilde{g}^{-1}\ (\widetilde{g}(E) - \{\widetilde{g}(E) \cap V\})$.

(2) $\widetilde{g}(E) \cap V$

$$= \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 < 1,\ y_1^2 + y_2^2 + y_3^2 \leqq \tfrac{1}{4},\ z = 0\}$$
$$\cup\ \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 \leqq \tfrac{1}{4},\ y_1^2 + y_2^2 + y_3^2 < 1,\ z = 0\}.$$

See Figure I.

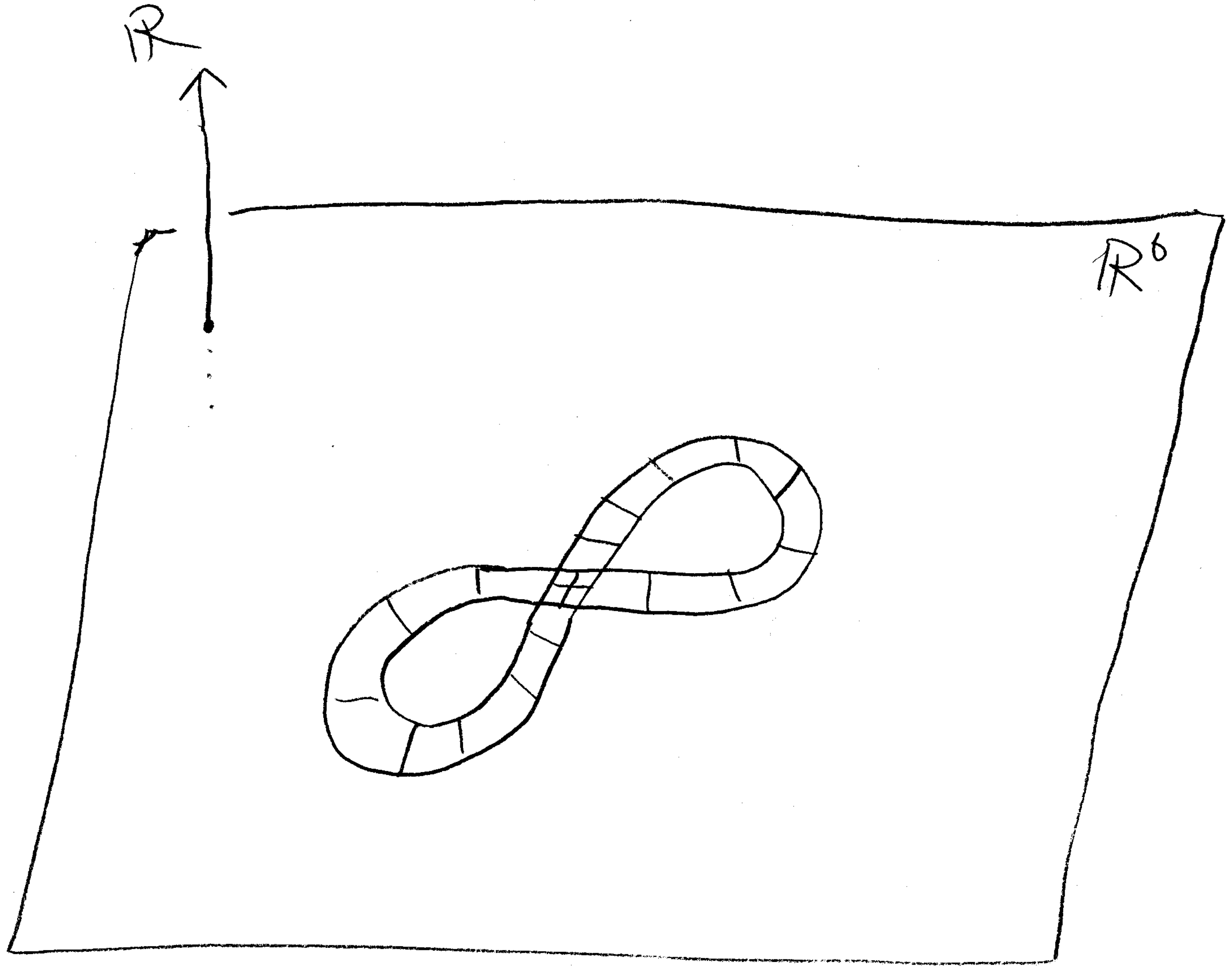

Figure 1

Let $f : E \hookrightarrow \mathbb{R}^6 \times \mathbb{R}^1$ be an embedding with the following properties.

(1) $\widetilde{g}|_{V^C} = f|_{V^C}$

(2) $f(E) \cap V$

$$= \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 < 1,\ y_1^2 + y_2^2 + y_3^2 \leqq \tfrac{1}{4},$$
$$z = 1 - (x_1^2 + x_2^2 + x_3^2)\ \}$$
$$\cup \{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 \leqq \tfrac{1}{4},\ y_1^2 + y_2^2 + y_3^2 < 1,\ z = 0\ \}.$$

We can make the corner smooth.

See Figure II.

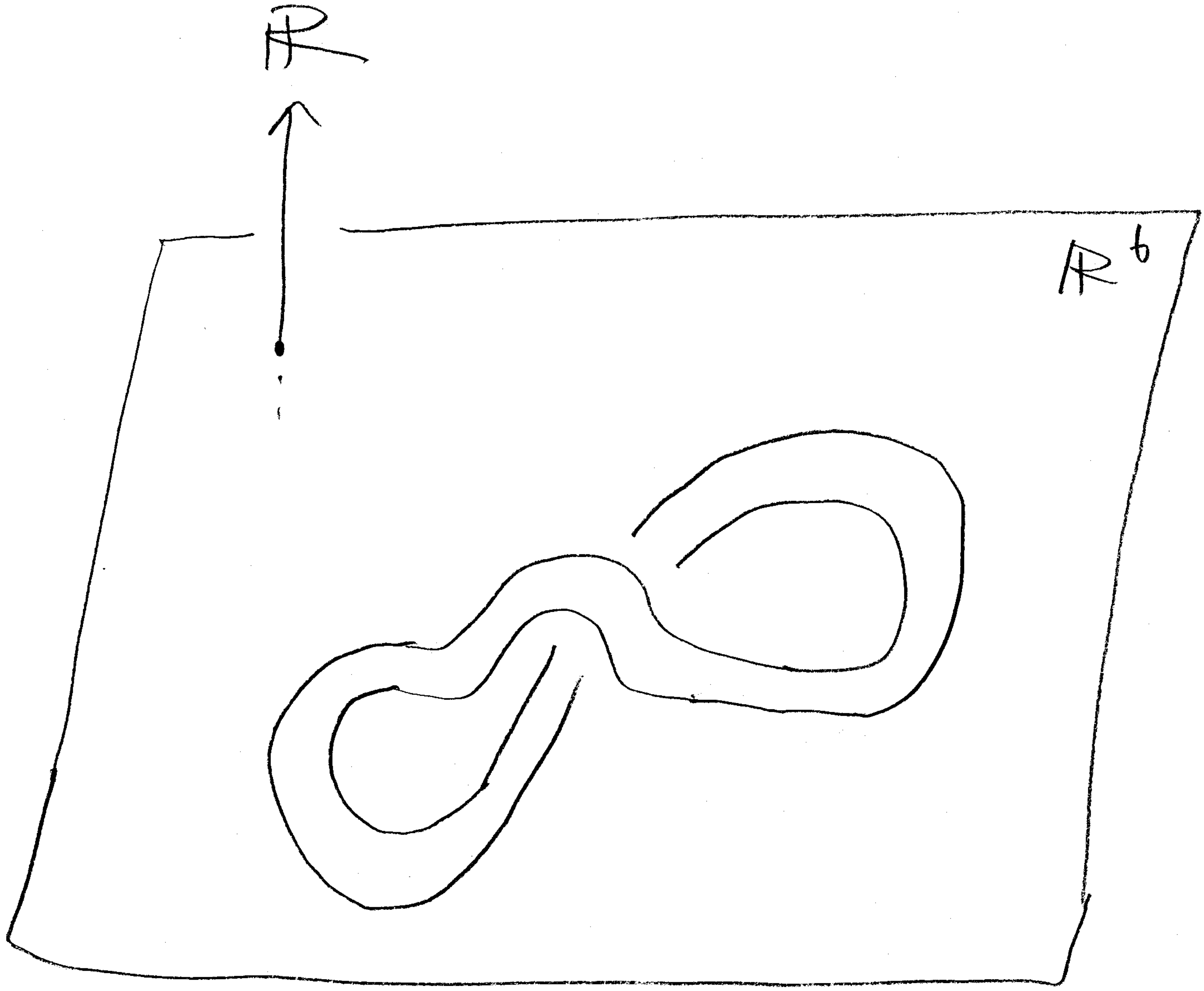

Figure 2

The submanifold $f(\partial E)$ is called K_1. Then the projection P_1 of K_1 is $\widetilde{g}(\partial E)$.

Then we have: The singular point set of the projection P_1 is $\{(x_1, x_2, x_3, y_1, y_2, y_3, z) |\ x_1^2 + x_2^2 + x_3^2 = \frac{1}{4},\ y_1^2 + y_2^2 + y_3^2 = \frac{1}{4},\ z = 0\}$. It consists of double points. It is diffeomorphic to $S^2 \times S^2$. It is connected. $\mu(P_1) = 1$.

We prove: K_0 is not equivalent to any of K_1, $-K_1$, K_1^* nor $-K_1^*$.

Proof. Let K be a codimension two submanifold of S^{n+2}. Let X_K denote the infinite cyclic covering space of the complement associated with the natural homomorphism map $\pi_1(S^{n+2} - K) \to H_1(S^{n+2} - K; \mathbb{Z}) \cong \mathbb{Z}$. We consider $H_*(X_K; \mathbb{Z})$ as a module over $\Lambda = \mathbb{Z}[t, t^{-1}]$. See [M], [L1] etc. for properties of such spaces and those of such modules.

We can regard that K_i is in $S^7 (i = 0, 1)$ naturally. We consider $H_3(X_{K_i}; \mathbb{Z})$. By the construction of K_i, we have:

(1) $H_3(X_{K_0}; \mathbb{Z}) \cong 0$.

(2) $H_3(X_{K_1}; \mathbb{Z}) \cong H_3(X_{-K_1}; \mathbb{Z}) \cong H_3(X_{K_1^*}; \mathbb{Z}) \cong H_3(X_{-K_1^*}; \mathbb{Z}) \cong \Lambda/(t-1) \cdot \Lambda$.

Therefore K_0 is equivalent to neither K_1, $-K_1$, K_1^* nor $-K_1^*$.

We next prove the case of $n > 5$.

We define an n-dimensional submanifold $K_i^{(n)} \subset \mathbb{R}^{n+2}$ as follows. ($n \geqq 5$, $i = 0, 1$.)

Let $K_i^{(5)}$ be K_i.

Put $\mathbb{R}^{n+2} = \{x | x \in \mathbb{R}\} \times \mathbb{R}^n \times \{t | t \in \mathbb{R}\}$. Suppose the projection map is $\mathbb{R}^{n+2} \to \{x | x \in \mathbb{R}\} \times \mathbb{R}^n \times \{t | t = 0\}$.

We assume $K_i^{(n)} \subset \{x | x \geqq 0\} \times \mathbb{R}^n \times \{t | t \in \mathbb{R}\}$ and $K_i^{(n)} \cap \{x | x = 0\} \times \mathbb{R}^n \times \{t | t \in \mathbb{R}\}$ is an n-disc.

We define $K_i^{(n+1)} \subset \mathbb{R}^{(n+3)}$ as follows. We consider $\mathbb{R}^{(n+3)} = \{(x, y) | x, y \in \mathbb{R}\} \times \mathbb{R}^n \times \{t | t \in \mathbb{R}\}$. We regard $\mathbb{R}^{(n+3)}$ as the result of rotating $\{x | x \geqq 0\} \times \mathbb{R}^n \times \{t | t \in \mathbb{R}\}$ around $\{x | x = 0\} \times \mathbb{R}^n \times \{t | t \in \mathbb{R}\}$. When rotating it, rotate $K_i^{(n)}$ as well. The result is called $K_i^{(n+1)}$.

By the construction of $K_i^{(n)}$, we have:

(1) For the projection of $P_i^{(n)}$ of $K_i^{(n)}$, $\mu(P_i^{(n)})$=1. The singular point set of P_i consists of double points.

(2) $H_3(X_{K_0^{(n)}}; \mathbb{Z}) \cong 0$.

(3) $H_3(X_{K_1^{(n)}}; \mathbb{Z}) \cong H_3(X_{-K_1^{(n)}}; \mathbb{Z}) \cong H_3(X_{K_1^{(n)*}}; \mathbb{Z}) \cong H_3(X_{-K_1^{(n)*}}; \mathbb{Z}) \cong \Lambda/(t-1) \cdot \Lambda$.

The computation for K_1 follows because the 0-section of E is a generator for $H_3(E)$ (Compare [Kf1], p.43, 190, 229).

Therefore $K_0^{(n)}$ is equivalent to neither $K_1^{(n)}$, $-K_1^{(n)}$, $K_1^{(n)*}$ nor $-K_1^{(n)*}$.

3. The proof of Theorem 1

We first prove the case of $n = 5$.

We use $f(E)$ in §2.

We suppose that $f(E) - V \subset \mathbb{R}^6 \times \{0\}$. Take a 6-ball $B^6 \subset \mathbb{R}^6 \times \{0\} \subset \mathbb{R}^6 \times \mathbb{R}$. In $B^6 \times \mathbb{R}$, take a submanifold A_1 which is a parallel displacement of the submanifold $f(E)$. In $(\mathbb{R}^6 - B^6) \times \mathbb{R}$, take a submanifold A_2 which is a parallel displacement of the submanifold $f(E)$ with the opposite orientation.

Recall $E = S^3 \times D^3$. We can put $A_i = S^3_i \times D^3 = (D^3_{iS} \cup D^3_{iN}) \times D^3 = (D^3_{iS} \times D^3) \cup (D^3_{iN} \times D^3)$ (i=1,2). Suppose $(D^3_{iS} \times D^3)$ is embedded in $\mathbb{R}^6 \times \{0\}$.

Take submanifolds S^2_1 and S^2_2 diffeomorphic to the 2-sphere in ∂B^6 so that the linking number is one.

See Figure III.

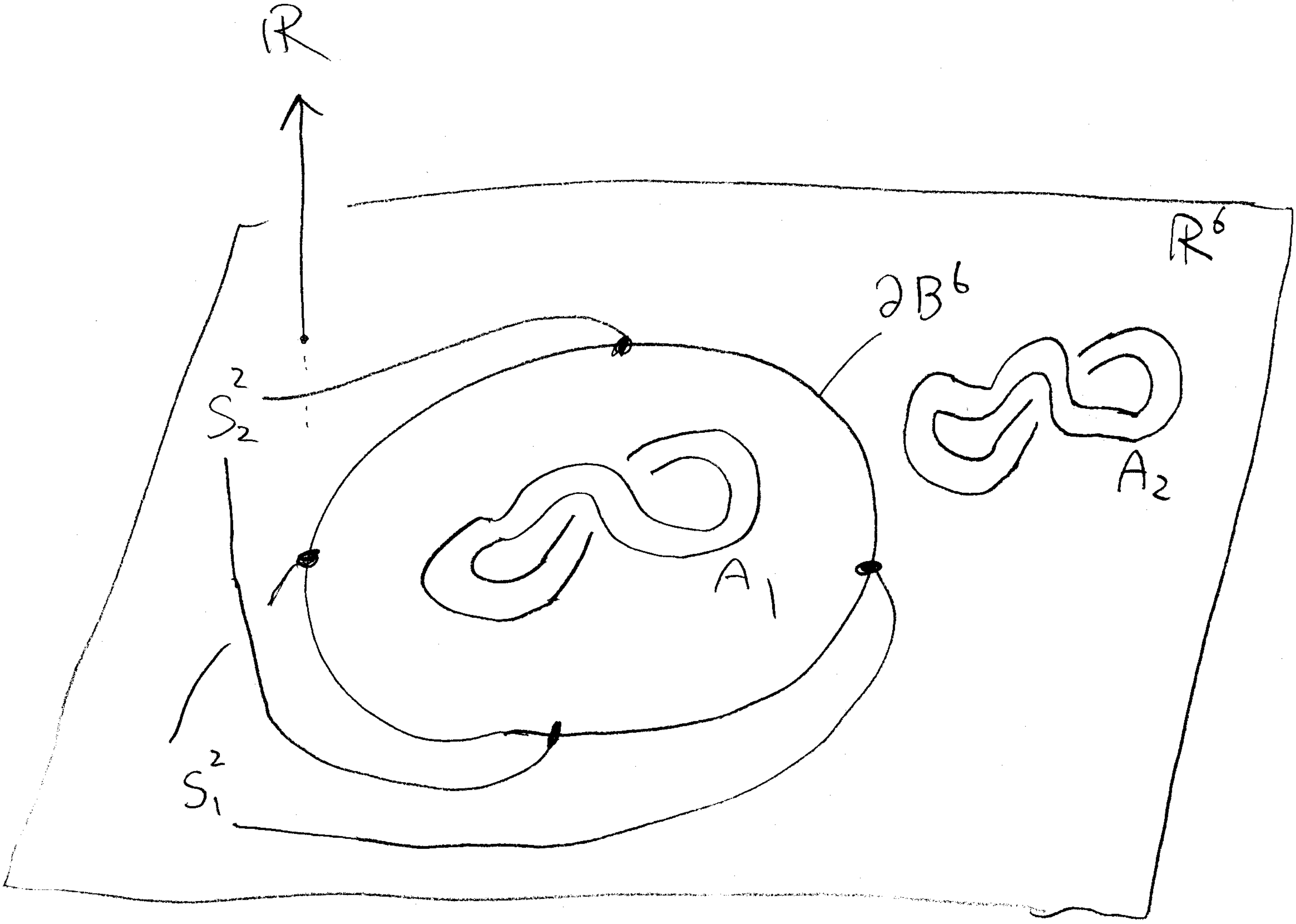

Figure 3

There are orientation preserving diffeomorphism maps h such that $h(\mathbb{R}^6 \times \{t\})=\mathbb{R}^6 \times \{t\}$.

By using such a diffeomorphism map, we move A_1 so that:

(1) $(\mathrm{Int} D^3_{1S}) \times D^3 \subset (\mathbb{R}^6 \times \{0\} - B^6)$

(2)$(\mathrm{Int} D^3_{1N}) \times D^3 \subset B^6 \times \mathbb{R}$. The singular point set of the projection of A_1 is in B^6.

(3)$A_1 \cap \partial B^6$=$(\partial D^3_{1S}) \times D^3$ =$(\partial D^3_{1N}) \times D^3$ and $\partial D^3_{1s} = \partial D^3_{1N} = S^2_1$.

By using such a diffeomorphism map, we move A_2 so that:

(1) $(\mathrm{Int} D^3_{2S}) \times D^3 \subset B^6$

(2)$(\mathrm{Int} D^3_{2N}) \times D^3 \subset (\mathbb{R}^6 - B^6) \times \mathbb{R}$. The singular point set of the projection of A_2 is in $\mathbb{R}^6 - B^6$.

(3)$A_2 \cap \partial B^6$=$(\partial D^3_{2S}) \times D^3$ =$(\partial D^3_{2N}) \times D^3$ and $\partial D^3_{2s} = \partial D^3_{2N} = S^2_2$.

We define K to be

$\overline{\partial B^6 - \partial(D^3_{1N} \times D^3) - \partial(D^3_{2N} \times D^3)} \cup \overline{[\partial(D^3_{1N} \times D^3) \cup \partial(D^3_{2N} \times D^3)] - \partial B^6}$.

See Figure IV.

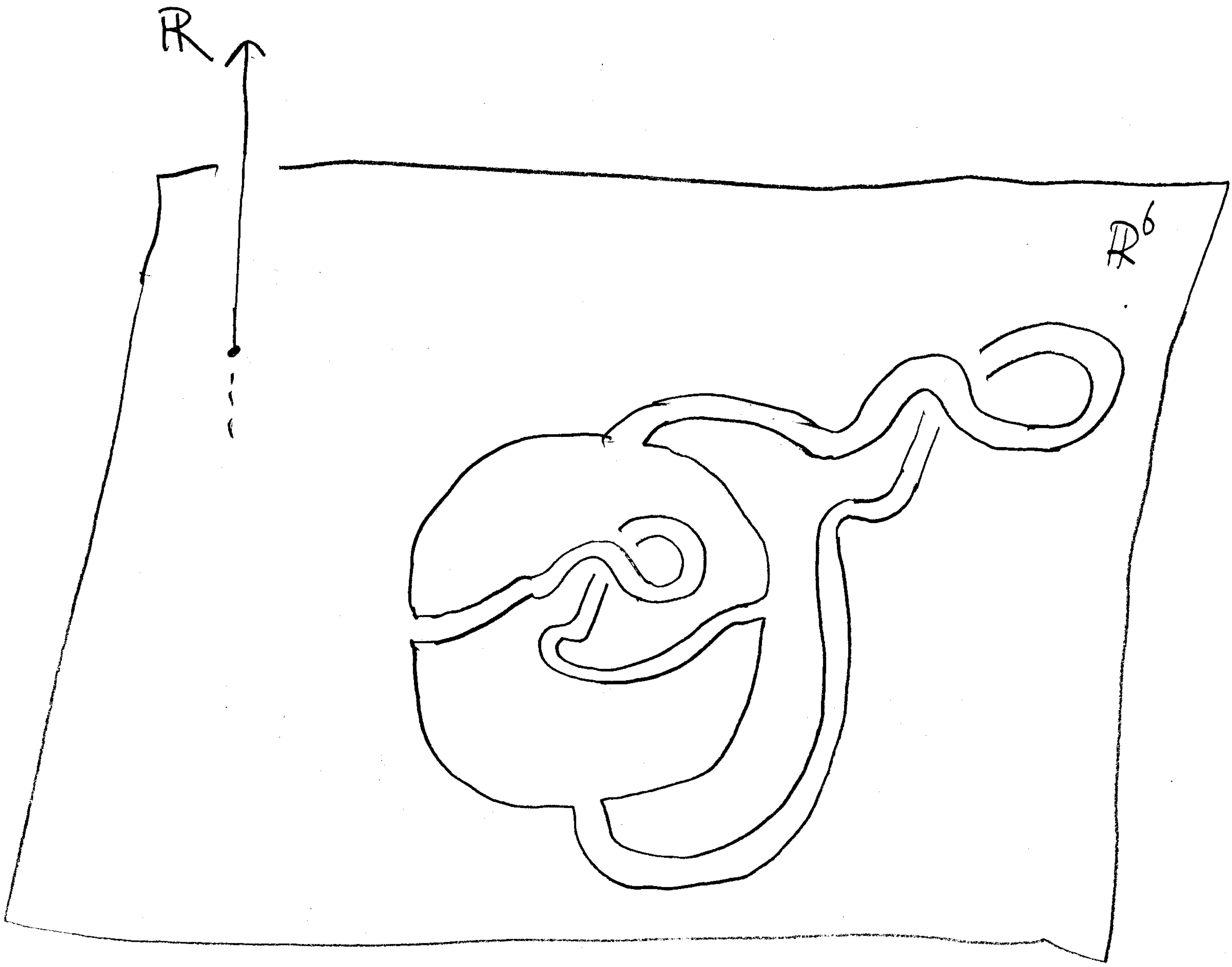

Figure 4

By the construction, we have:

(1) K is diffeomorphic to the 5-sphere. K is a 5-knot.

(2) For the projection P of K, $\mu(P)$=2. The singular point set of P consists of double points.

(3) A Seifert matrix of K is $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. (See [L1] and [L2] for Seifert matrices.) Hence $H_3(X_K; \mathbb{Z}) \cong \Lambda/(t^2 - 3t + 1) \cdot \Lambda$. Therefore K is truly knotted.

This completes the proof in the case of $n = 5$.

We next prove the case of $n > 5$.

Let $K^{(5)}$ be K. Let $K^{(n+1)}$ be the spun knot of $K^{(n)}(n \geqq 5)$. (See [Z] for spun knots.)

We take the axis as in the proof of the $n > 5$ case in §2. Then the projection $P^{(n+1)}$ of $K^{(n+1)}$is the result of rotating $P^{(n)}$ around the axis. Hence $\mu(P)$=2. The singular point set of P consists of double points.

By the construction, we have $H_3(X_{K^{(n+1)}}; \mathbb{Z}) \cong \Lambda/(t^2 - 3t + 1) \cdot \Lambda$. Therefore $K^{(n+1)}$ is truly knotted.

This completes the proof.

4. The proof of Theorem 2

We use K, $K^{(n)}$, and $P^{(n)}$ in §3.

We first prove the case of $n = 5$.

Let K' be a 5-knot. Suppose that the projection of K' is the projection P of K.

Then a Seifert matrix of K' is one of the following.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Hence $H_3(X_{K'}; \mathbb{Z})$ is not trivial. Therefore K' is truly knotted.

We next prove the case of $n > 5$.

Let $K^{(n)'}$ be an n-knot($n > 5$). Suppose the projection of $K^{(n)'}$ is the projection $P^{(n)}$ of $K^{(n)}$. Then $K^{(n)'}$ is a spun knot of an $(n-1)$-knot whose projection is $P^{(n-1)}$. Hence $H_3(X_{K^{(n)'}}; \mathbb{Z})$ is not trivial. Therefore $K^{(n)'}$ is truly knotted.

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