Intersectional pairs of n-knots, local moves of n-knots, and their associated invariants of n-knots

Eiji Ogasa

pqr100pqr100@yahoo.co.jp

Abstract. Let n be an integer>0. Let S_1^{n+2} (respectively, S_2^{n+2}) be the $(n+2)$-sphere embedded in the$(n+4)$-sphere S^{n+4}. Let S_1^{n+2} and S_2^{n+2} intersect transversely. Suppose that the smooth submanifold $S_1^{n+2} \cap S_2^{n+2}$ in S_i^{n+2} is PL homeomophic to the n-sphere. Then $S_1^{n+2} \cap S_2^{n+2}$ in S_i^{n+2} is an n-knot K_i. We say that the pair (K_1, K_2) of n-knots is realizable.

We consider the following problem in this paper. Let A_1 and A_2 be n-knots. Is the pair (A_1, A_2) of n-knots realizable?

We give a complete characterization.

Chapter I is accepted in
Mathematical Research Letters, 1998, 5, 577-582.
This manuscript is not the published version.

Chapter I is a summary of Chapter II. Chapter II is beased on the author's PhD thesis (University of Tokyo, 1996) University of Tokyo preprint series UTMS 95-50 is a preprint of Chapter II.

INTERSECTIONAL PAIRS OF n-KNOTS, LOCAL MOVES OF n-KNOTS, AND THEIR ASSOCIATED INVARIANTS OF n-KNOTS

Eiji Ogasa
ogasa@ms.u-tokyo.ac.jp, ogasa@max.math.brandeis.edu

1. Introduction

Our first purpose is to discuss the following problem.

Let S_1^{n+2} and S_2^{n+2} be $(n+2)$-spheres embedded in the $(n+4)$-sphere S^{n+4} $(n \geqq 1)$ which intersect transversely. If we assume $M = S_1^{n+2} \cap S_2^{n+2}$ is PL homeomorphic to the single standard n-sphere, we obtain a pair of n-knots, M in S_1^{n+2} and M in S_2^{n+2}. We consider which pairs of n-knots we obtain as above. That is, let (K_1, K_2) be a pair of n-knots. Then we consider whether the pair of n-knots (K_1, K_2) is obtained as above.

We give a complete answer to this problem (Theorem 3.1).

In order to get the complete answer, we introduce a local move of n-knots $(n \geqq 1)$. Furthermore, we show a relation between the local move and some invariants of n-knots (Theorem 4.1 and Corollary 4.2).

Our second purpose is to discuss the relation between the local move and the invariants of n-knots. In the case of 1-links, there is a great deal known about relations between local moves and knot invariants. (See e.g. [V][Wi][Ka2].) Our discussion is a high dimensional version of this theory.

2. Definitions

An *(oriented) (ordered) m-component n-(dimensional) link* is a smooth, oriented submanifold $L = \{K_1, ..., K_m\}$ of S^{n+2}, which is the ordered disjoint union of m connected oriented

This research was partially supported by Research Fellowships of the Promotion of Science for Young Scientists.

submanifolds, each PL homeomorphic to the standard n-sphere. If $m = 1$, then L is called a *knot*. (This definition is used often. See e.g. [CO],[L1],[L3].)

Let L_1 and L_2 be n-links. L_1 is said to be equivalent to L_2 if there exists an orientation preserving diffeomorphism h of S^{n+2} such that $h|L_1$ is an orientation preserving diffeomorphism from L_1 to L_2.

We work in the smooth category.

Definition (K_1, K_2) is called a *pair of n-knots* if K_1 and K_2 are n-knots. (K_1, K_2, X_1, X_2) is called a *4-tuple of n-knots and $(n+2)$-knots* or a *4-tuple of $(n, n+2)$-knots* if (K_1, K_2) is a pair of n-knots and X_1 and X_2 are $(n+2)$-knots diffeomorphic to the standard $(n+2)$-sphere. $(n \geqq 1)$.

Definition. A 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ satisfying the following conditions. $(n \geqq 1)$.

(1) The intersection $\Sigma = f(S_1^{n+2}) \cap f(S_2^{n+2})$ is PL homeomorphic to the standard n-sphere.
(2) $f^{-1}(\Sigma)$ in S_i^{n+2} defines an n-knot K_i $(i = 1, 2)$.
(3) $f|S_i^{n+2}$ is an embedding. $f(S_i^{n+2})$ in S^{n+4} is equivalent to X_i (i=1,2).

A pair of n-knots (K_1, K_2) is said to be *realizable* or is called an *intersectional pair of n-knots* if there is a realizable 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2).

3. Intersectional pair of n-knots

Our main theorem is:

Theorem 3.1. *A pair of n-knots (K_1, K_2) $(n \geqq 1)$ is realizable if and only if (K_1, K_2) satisfies the condition that*

$$\begin{cases} (K_1, K_2) \text{ is arbitrary} & \text{if } n \text{ is even,} \\ \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \text{if } n = 4m+1, \quad (m \geqq 0). \\ \sigma(K_1) = \sigma(K_2) & \text{if } n = 4m+3, \end{cases}$$

There is a mod 4 periodicity in dimension. It is similar to the periodicity in knot cobordism theory ([L1]) and surgery theory (See e.g. [Br][Wa][CS][We]).

We have the following result on the realization of 4-tuples of $(n, n+2)$-knots.

Theorem 3.2. *A 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable if K_1 and K_2 are slice. $(n \geqq 1)$. In particular, if n is even, an arbitrary 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable.*

Note. (1) Kervaire proved that all even dimensional knots are slice ([Ke]).

(2) In [O1] the author discussed the case of two 3-spheres in a 5-sphere. In [O2] the author discussed the case of the intersection of three 4-spheres in a 6-sphere.

Problem. Which 4-tuples of $(2n+1, 2n+3)$-knots are realizable $(n \geqq 1)$?

4. High-dimensional pass-moves

In order to prove Theorem 3.1, we introduce a new local move for high dimensional knots, the *high dimensional pass-move.* Pass-moves for 1-knots are discussed in p.146 of [Ka].

We define high dimensional pass-moves for $(2k+1)$-knots $\subset S^{2k+3}$.

Definition Take a trivially embedded $(2k+3)$-ball $B = B^{2k+2}\times[-1,1]$ in S^{2k+3}. We define $J_+, J_- \subset B$ as follows. Refer to Figure 4.1.

In $\partial B^{2k+2}\times\{0\}$, take trivially embedded S^k_1, S^k_2 such that $\mathrm{lk}(S^k_1, S^k_2)=1$. Let $N(S^k_*)$ be a tubular neighborhood of S^k_* in $\partial B^{2k+2}\times\{0\}$.

Let h^{k+1} be an $(2k+2)$-dimensional $(k+1)$-handle which is attached to $\partial B^{2k+2}\times\{0\}$ along $N(S^k_1)$ with the trivial framing and which is embedded trivially in $B^{2k+2}\times\{0\}$.

Let h^{k+1}_+ (resp. h^{k+1}_-) be an $(2k+2)$-dimensional $(k+1)$-handle which is embedded in $B = B^{2k+2}\times[0,1]$ (resp. $B = B^{2k+2}\times[-1,0]$) and which is attached to $\partial B^{2k+2}\times\{0\}$ along $N(S^k_2)$ with the trivial framing.

Let $h^{k+1}_+ \cap h^{k+1}_- = N(S^k_2)$. Let $h^{k+1}_+ \cap h^{k+1} = h^{k+1}_- \cap h^{k+1} = \phi$.

Let J_+ be a submanifold $\overline{(\partial h^{k+1}) - N(S^k_1)} \amalg \overline{(\partial h^{k+1}_+) - N(S^k_2)}$ in B.

Let J_- be a submanifold $\overline{(\partial h^{k+1}) - N(S^k_1)} \amalg \overline{(\partial h^{k+1}_-) - N(S^k_2)}$ in B.

In Figure 4.1, we draw $B = B^{2k+2}\times[-1,1]$ by using the projection to $B^{2k+2}\times\{0\}$.

Figure4.1.Seethelastpage.

Let K_+, K_- be $(2k+1)$-knots $\subset S^{2k+3}$. We say that K_+ is obtained from K_- by one *high dimensional pass-move* if there is a trivially embedded $(2k+2)$-ball $B \subset S^{2k+3}$ such that $K_+\cap B$ is J_+ and $K_-\cap B$ is J_-.

Let K, K' be $(2k+1)$-knots $\subset S^{2k+3}$. We say that K is *pass-move equivalent* to K' if there are $(2k+1)$-knots K_1,...,K_μ ($\mu\in\mathbb{N}$) such that K_i is pass-move equivalent to K_{i+1}.

We prove:

Theorem 4.1. *For $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. ($k \geqq 1$.)*

(1) *There exists a $(2k+1)$-knot K_3 which is pass-move equivalent to K_1 and cobordant to K_2.*

(2) *K_1 and K_2 satisfy the condition that* $\begin{cases} \mathrm{Arf}(K_1)=\mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.}\end{cases}$

The $k=0$ case of Theorem 4.1 follows from [Ka].

Corollary 4.2. *Let K_1 and K_2 be $(2k+1)$-knots ($k \geqq 1$.). Suppose that K_1 is pass-move equivalent to K_2. Then K_1 and K_2 satisfy the condition that*

$$\begin{cases} \mathrm{Arf}(K_1)=\mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.}\end{cases}$$

Note. In [O3] the author proved a relation between another local move of 2-knots and other invariants of 2-knots.

5. Proof of Theorem 3.1

We prove the following lemmas by explicit construction.

Lemma 5.1. *Let K be an n-knot. Then the pair of n-knots (K, K) is realizable ($n \geqq 1$).*

Lemma 5.2. *Let K_1 and K_2 be $(2k+1)$-knots. Suppose that K_1 is pass-move equivalent to K_2. Then the pair of $(2k+1)$-knots (K_1, K_2) is realizable ($k \geqq 0$).*

Lemma 5.3. *Let K_1, K_2 and K_3 be n-knots ($n \geqq 1$). Suppose that the pair of n-knots (K_1, K_2) is realizable and that K_2 is cobordant to K_3. Then the pair of n-knots (K_1, K_3) is realizable.*

Theorem 3.1 is deduced from Theorem 4.1 and Lemmas 5.1, 5.2, 5.3.

6. Proof of Theorem 3.2

It suffices to prove that a 4-tuple of $(n, n+2)$-knots (K_1, K_2, T, T) is realizable, where K_1 is a slice n-knot, K_2 is the trivial n-knot, T is the trivial $(n+2)$-knot.

Any 1-twist spun knot is unknotted ([Z]). Theorem 3.2 follows from this fact.

7. The proof of Theorem 4.1

Every p-knot ($p > 1$) is cobordant to a simple knot. (See [L1] for a proof and the definition of simple knots.) By using this fact, we prove that the $k \geqq 1$ case of Theorem 4.1 can be deduced from Theorem 7.1.

Proposition 7.1. *For simple $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. ($k \geqq 1$.)*

(1) *K_1 is pass-move equivalent to K_2.*

(2) *K_1 and K_2 satisfy the condition that* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

Proof of Proposition 7.1. (2)⇒(1). K_1 bounds a Seifert hypersurface V_1 with a handle decomposition (one 0-handle)∪((k+1)-handles). Take a Seifert matrix associated with V_1. By using high dimensional pass moves, we can change the Seifert matrix without changing the diffeomorphism type of V_1. Thus we obtain a $(2k+1)$-knot K_2' whose Seifert matrix is same as the Seifert matrix of K_2 if (2) holds. By the classification theorem of simple knots by [L2], K_2' is equivalent to K_2.

(1)⇒(2). Suppose that $(2k+1)$-knots $K_* \subset S_*^{2k+3}$ bounds a Seifert hyper surface V_*. Note V_* are $(2k+2)$-manifolds. There is a compact oriented parallelizable $(2k+4)$-manifold P whose boundary is $S_1^{4k+3} \amalg S_2^{4k+3}$ containning compact oriented $(2k+3)$-manifold Q whose boundary is $V_1 \cup (S^{2k+1} \times [1, 2]) \cup V_2$. (Here, ∂V_* is K_* and $S^{2k+1} \times \{*\}$ is K_*.) We use characteristic classes and intersection products to prove (1)⇒(2).

8. Intersectional pair of submanifolds

In §1 suppose M is not PL homeomorphic to the standard sphere. Then we obtain a pair of submanifolds, M in S_i^{n+2} ($i = 1, 2$).

Let N be a closed oriented manifold. (K_1, K_2) is called *a pair of submanifolds (diffeomorphic to N)* if K_i is a submanifold of S^{n+2} diffeomorphic to N.

Let (K_1, K_2) be a pair of submanifolds diffeomorphic to M. We say (K_1, K_2) is an *intersectional pair* if the submanifold K_i is equivalent to the submanifold $M = S_1^{n+2} \cap S_2^{n+2}$ in S_i^{n+2} as in §1 $(i = 1, 2)$.

It is natural to ask the following problem.

Problem 8.1. Which pairs of submanifolds are intersectional pairs?

The author can prove the following results.

When n is even, not all pair of submanifolds as above are realizable.

When $n = 4m + 3$, we can define the signature as in the knot case and the signature is an obstruction. Therefore not all pairs are realizable. When $n = 3$, (K_1, K_2) is realizable if and only if $\sigma(K_1)= \sigma(K_2)$. When $n \neq 3$, $\sigma(K_1)= \sigma(K_2)$ does not imply (K_1, K_2) is realizable in general.

When $n = 4m + 1$, there is a closed oriented manifold M such that if K_1 and K_2 are PL homeomorphic to M, then (K_1, K_2) is realizable. In other words, there is no invariant corresponding to the Arf invariant as in the knot case. Of course, not all pairs are realizable.

References

[Br] W. Browder, *Surgery on simply-connected manifolds*, Springer (1972).

[CS] Cappell and Shaneson, *The codimension two placement problem and homology equivalent manifolds*, Ann of Math **99** (1974).

[CO] T. Cochran and K. Orr, *Not all links are concordant to boundary links*, Ann. Math. **138** (1993), 519–554.

[Ka] L. Kauffman, *Formal knot theory*, Princeton University Press Math. Notes **30** (1983).

[Ka2] L. Kauffman, *Knots and Physics*, World Scientific, Series on Knots and Everything **1** (1991).

[Ke] M. Kervaire, *Les noeudes de dimensions supéreures*, Bull.Soc.Math.Fr. **93** (1965), 225-271.

[L1] J. Levine, *Knot cobordism in codimension two*, Comment. Math. Helv. **44** (1969), 229-244.

[L2] J.Levine, *An algebraic classification of some knots of codimension two*, Comment.Math.Helv. **45** (1970), 185-198.

[L3] J.Levine, *Link invariants via the eta-invariant*, Comment.Math.Helv. **69** (1994), 82-119.

[O1] E. Ogasa, *The intersection of spheres in a sphere and a new geometric meaning of the Arf invariant*, University of Tokyo Preprint.

[O2] E. Ogasa, *The intersection of three spheres in a sphere and a new application of the Sato-Levine invariant*, Proc AMS (To appear).

[O3] E. Ogasa, *Ribbon moves of 2-links: The μ-invariants of 2-links and $TorH_1(\; ; \;)$ of Seiferthypersurfaces*, Preprint.

[V] V. A. Vassiliev, *Complements of Discriminants of smooth maps: Topology and Applications*, Translations of Mathematical Monographs, American Mathematical Society **98** (1994).

[Wa] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press **New York and London** (1970).

[Wi] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989), 351-399.

[We] S. Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Math. (1994).

Acknowledgement. The author would like to thank Prof. Levine for his interest in this paper and correcting the author's English. The author would like to thank the referee and the editor for their reading with patience.

Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153, Japan
Department of Mathematics, Brandeis University, Waltham, MA02254, USA

ON THE INTERSECTION OF SPHERES IN A SPHERE II: HIGH DIMENSIONAL CASE

Eiji Ogasa

Abstract. Consider transverse immersions $f : S_1^{n+2} \amalg S_2^{n+2} \looparrowright S^{n+4}$ such that $f|S_i^{n+2}$ is an embedding and the intersection $f(S_1^{n+2}) \cap f(S_2^{n+2})$ is PL homeomorphic to the standard n-sphere. ($n \geqq 1$). Then we obtain a pair of n-knots, $f^{-1}(f(S_1^{n+2}) \cap f(S_2^{n+2}))$ in S_i^{n+2} ($i = 1, 2$). We determine which pair of n-knots are obtained as above. Roughly speaking, our result is characterized by the Arf invariant and the signature. We find a mod 4 periodicity in the dimension n.

1.Introduction and Main results

Let S_1^{n+2} and S_2^{n+2} be the $(n+2)$-spheres embedded in the $(n+4)$-sphere S^{n+4} ($n \geqq 1$) and intersect transversely. Here, the orientation of the intersection M is induced naturally. If we assume M is PL homeomorphic to the single standard n-sphere, we obtain a pair of n-knots, M in S_i^{n+2} ($i = 1, 2$), and a pair of $(n+2)$-knots, S_i^{n+2} in S^{n+4} ($i = 1, 2$).

Conversely, let (K_1, K_2) be a pair of n-knots. It is natural to ask whether (K_1, K_2) is obtained as above. In this paper we give a complete answer to the above question. Furthermore we discuss somewhat which 4-tuple(K_1, K_2, S_1^{n+2}, S_2^{n+2}) are realizable.

To state our results we need some definitions.

An *(oriented) (ordered) m-component n-(dimensional) link* is a smooth, oriented submanifold $L = \{K_1, ..., K_m\}$ of S^{n+2}, which is the ordered disjoint union of m manifolds, each

This research was partially supported by Research Fellowships of the Promotion of Science for Young Scientists.

This research was partially suppported by Reseach Fellowships of thePromotion of Science for Young Scientists.

PL homeomorphic to the standard n-sphere. (If $m = 1$, then L is called a *knot.*) We say that m-component n-dimensional links, L_0 and L_1, are said to be *(link-)concordant* or *(link-)cobordant* if there is a smooth oriented submanifold $\widetilde{C}=\{C_1 ,...,C_m\}$ of $S^{n+2} \times [0,1]$, which meets the boundary transversely in $\partial\widetilde{C}$, is PL homeomorphic to $L_0 \times [0,1]$ and meets $S^{n+2} \times \{l\}$ in L_l $(l = 0,1)$. (See [CO]).

Definition 1.1. (K_1, K_2) is called a *pair of n-knots* if K_1 and K_2 are n-knots. (K_1, K_2, X_1, X_2) is called a *4-tuple of n-knots and $(n+2)$-knots* or a *4-tuple of $(n, n+2)$-knots* if K_1 and K_2 compose a pair of n-knots (K_1, K_2) and X_1 and X_2 are $(n+2)$-knots diffeomorphic to the standard $(n+2)$-sphere. $(n \geqq 1)$.

Definition 1.2. A 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ satisfying the following conditions. $(n \geqq 1)$.

(1) $f|S_i^{n+2}$ defines X_i (i=1,2).
(2) The intersection $\Sigma = f(S_1^{n+2}) \cap f(S_2^{n+2})$ is PL homeomorphic to the standard n-sphere.
(3) $f^{-1}(\Sigma)$ in S_i^{n+2} defines an n-knot K_i $(i = 1,2)$.

A pair of n-knots (K_1, K_2) is said to be *realizable* if there is a realizable 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2). Then f is called an *immersion to realize* (K_1, K_2, X_1, X_2) or (K_1, K_2).

The following theorem characterizes the realizable pair of n-knots.

Theorem 1.3. *A pair of n-knots (K_1, K_2) $(n \geqq 1)$ is realizable if and only if (K_1, K_2) satisfies the condition that*

$$\begin{cases} (K_1, K_2) \text{ is arbitrary} & \text{if } n \text{ is even,} \\ \mathrm{Arf}(K_1)=\mathrm{Arf}(K_2) & \text{if } n = 4m+1, \quad (m \geqq 0, m \in \mathbb{Z}). \\ \sigma(K_1)=\sigma(K_2) & \text{if } n = 4m+3, \end{cases}$$

There exists a mod 4 periodicity in dimension. It is similar to the periodicity in the knot cobordism theory and the surgery theory.

We have the following results on the realization of 4-tuple of $(n, n+2)$-knots.

Theorem 1.4. *A 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable if K_1 and K_2 are slice. $(n \geqq 1)$.*

Kervaire proved that all even dimensional knots are slice ([K]). Hence we have:

Corollary 1.5. *If n is even, an arbitrary 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable.*

The author discussed related topics in [O1], [O2], and [O3].

This paper is organized as follows. In §2 we introduce a new knotting operation, the high dimensional pass-move, and state its relation to the Arf invariant and the signature. In §3 we discuss a sufficient condition for the realization of pair of odd dimensional knots. In §4 we discuss a necessary condition for the realization of pair of $(4m+1)$-knots. In §5 we discuss a necessary condition for the realization of pair of $(4m+3)$-knots. In §6 we prove Theorem 1.4 which induces a necessary and sufficient condition for the realization of pair of even dimensional knots. Theorem 1.3 follows from §2-6.

The author would like to thank Professor Takashi Tsuboi for his advise.

2. High dimensional pass-moves

In this section we introduce a new knotting operation for high dimensional knots. The 1-dimensional case of Definition 2.1 is disscussed in P. 146 of [Kf].

Definition 2.1. Take a $(2k+1)$-knot K ($k \geqq 0$). Let K be defined by a smooth embedding $g : \Sigma^{2k+1} \hookrightarrow S^{2k+3}$, where Σ^{2k+1} is PL homeomorphic to the standard $(2k+1)$-sphere. Let D_x^{k+1}=$\{(x_1, ..., x_{k+1})| \ \Sigma x_i^2 < 1\}$ and D_y^{k+1}=$\{(y_1, ..., y_{k+1})| \ \Sigma y_i^2 < 1\}$. Let $D_x^{k+1}(r)$=$\{(x_1, ..., x_{k+1})| \ \Sigma x_i^2 \leqq r^2\}$ and $D_y^{k+1}(r)$=$\{(y_1, ..., y_{k+1})| \ \Sigma y_i^2 \leqq r^2\}$. A local chart (U, ϕ) of S^{2k+3} is called a *pass-move-chart of* K if it satisfies the following conditions.

(1) $\phi(U) \cong \mathbb{R}^{2k+3} = (0,1) \times D_x^{k+1} \times D_y^{k+1}$
(2) $\phi(g(\Sigma^{2k+1}) \cap U) = [\{\frac{1}{2}\} \times D_x^{k+1} \times \partial D_y^{k+1}(\frac{1}{3})] \amalg [\{\frac{2}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times D_y^{k+1}]$

Let $g_U : \Sigma^{2k+1} \hookrightarrow S^{2k+3}$ be an embedding such that:

(1) $g|\{\Sigma^{2k+1} - g^{-1}(U)\} = g_U|\{\Sigma^{2k+1} - g^{-1}(U)\}$, and
(2) $\phi(g_U(\Sigma^{2k+1}) \cap U) = [\{\frac{1}{2}\} \times D_x^{k+1} \times \partial D_y^{k+1}(\frac{1}{3})] \amalg$

$$[\{\tfrac{2}{3}\} \times \partial D_x^{k+1}(\tfrac{1}{3}) \times (D_y^{k+1} - D_y^{k+1}(\tfrac{1}{2}))]$$
$$\cup [[\tfrac{1}{3}, \tfrac{2}{3}] \times \partial D_x^{k+1}(\tfrac{1}{3}) \times \partial D_y^{k+1}(\tfrac{1}{2})]$$
$$\cup [\{\tfrac{1}{3}\} \times \partial D_x^{k+1}(\tfrac{1}{3}) \times D_y^{k+1}(\tfrac{1}{2})\}]$$

Let K_U be the $(2k+1)$-knot defined by g_U. Then we say that K_U is obtained from K by the *(high dimensional) pass-move in* U. We say that $(2k+1)$-knot K and K' are *(high dimensional) pass-move equivalent* if there exist $(2k+1)$-knots K=K_1,K_2...,K_q,K_{q+1}=K' and K_{i+1} is obtained from K_i by the high dimensional pass-move in a pass-move-chart of K_i $(i = 1, ..., q)$.

High dimensional pass-moves have the follwing relation with the Arf invariant and the signature of knots. (The case of $k = 0$ follows from [Kf].) We prove:

Theorem 2.2. *For $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. ($k \geqq 0$.)*

(1) *There exists a $(2k+1)$-knot K_3 which is pass-move equivalent to K_1 and cobordant to K_2.*
(2) *K_1 and K_2 satisfy the condition* $\begin{cases} Arf(K_1) = Arf(K_2) & \text{when } k \text{ is even} \\ \sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd.} \end{cases}$

Organization of the proof of Theorem 2.2 is as follows. Obviously Theorem 2.2 is equivalent to the following Claim 2.2.1 and 2.2.2.

Claim 2.2.1. *If (2) of Theorem 2.2 holds, then (1) of Theorem 2.2 holds.*

Claim 2.2.2. *If (1) of Theorem 2.2 holds, then (2) of Theorem 2.2 holds.*

In this section we prove Claim 2.2.1. (We use Claim 2.2.1 in §3.) We use the results of §3, 4 and 5 and prove Claim 2.2.2. (Note. We don't use Claim 2.2.2 in the proof of §3, 4 and 5.) The proof of Claim 2.2.2 is written in §5.A. after §5.

We begin the proof of Claim 2.2.1. We need the following Lemma 2.3. We prove:

Lemma 2.3. *If a $(2k+1)$-knot K $(k \geqq 0)$ satisfy the condition*
$(*)\begin{cases} \text{Arf}(K)=0 & \text{when } k \text{ is even} \\ \sigma(K)=0 & \text{when } k \text{ is odd,} \end{cases}$ *then there exists a $(2k+1)$-knot $\widetilde{K}$ which is pass-move equivalent to the trivial knot and cobordant to the $(2k+1)$-knot K.*

Before proving Lemma 2.3, We prove:

Claim. Lemma 2.3 induces Claim 2.2.1.

Proof. $(-K_1^*)\sharp K_2$ satisfies the condition $(*)$. By Lemma 2.3, there exists a $(2k+1)$ knot $\widetilde{K}$ which is pass-move equivalent to the trivial knot and cobordant to $(-K_1^*)\sharp K_2$ Define K_3 to be $K_1\sharp\ \widetilde{K}$. Then the following (1) and (2) hold. (1)K_3 is pass-move equivalent to K_1. (2)$K_3 = K_1\sharp\widetilde{K}$ is cobordant to $K_1\sharp(-K_1^*)\sharp K_2$ and to K_2. □

Before proving Lemma 2.3, we review some definitions. (See [L2] and [Kw] for detail.) We first review on the definition of the Seifert matrix. Let K be a $(2k+1)$-knot and F a Seifert hypersurface. Let $F\times[-1,1]$ be embedded in S^{2k+3} so that $F\times\{0\}$ coincides with F and the standard orientation of [-1,1] coincides with the orientation of the normal bundle induced from that of F and that of S^{2k+3}. For $(k+1)$-cycles u and v in F, define $\theta(u,v)$ to be $lk(u, v\times\{1\})$ in S^{2k+3}. Let $z_1, ..., z_p$ be $(k+1)$-cycles in F which represent basis of $H_{k+1}(F:\mathbb{Z})/$(Torsion part). Define the *Seifert matrix* $A=\{a_{ij}\}$ *of* K *associated with* F *and* z_i to be $a_{ij}=\theta(z_i,z_j)$. Here, recall the following lemma 2.4.

Lemma 2.4. *(Well-known.)*

(1) *Let K be a $(2k+1)$-knot $(k\geqq 0)$ with a Seifert hypersurface F. Let u and v be $(k+1)$-cycles in $F\subset S^{2k+3}$. We have:*
$$\theta(u,v)+(-1)^{k+1}\theta(v,u)=u\cdot v,$$
where $u\cdot v$ is the intersection number in F.

(2) *For vanishing $(k+1)$-cycles μ and ν in S^{2k+3} $(k\geqq 0)$, where $\mu\cap\nu=\phi$,*
$$lk(\mu,\nu)=(-1)^k lk(\nu,\mu).$$

(3) *Let K be a $(2k+1)$-knot with a Seifert matrix A $(k\geqq 0)$. For appropriate Seifert surfaces, the Seifert matrixes of $-K$, that of K^* and that of $-K^*$ are $(-1)^k\{{}^tA\}$, $(-1)^{(k+1)}\{{}^tA\}$ and $-A$, respectively.*

Recall the following theorem.

Theorem 2.5. *([L1]) (2k+1)-knots $(k\geqq 1,\ k\neq 0)$ K_1 and K_2 are cobordant if and only if for a Seifert matrix A_i of $K_i (i=1,2)$, $\begin{pmatrix} A_1 & O \\ O & -A_2\end{pmatrix}$ is congruent to $\begin{pmatrix} O & N_1 \\ N_2 & N_3\end{pmatrix}$, where N_i are same size.*

We next review on the definition of the Arf invariant. Let K be a $(4m+1)$-knot. For $(2m+1)$-cycles x in F, define $q(x)\in\mathbb{Z}_2$ to be $\theta(x,x)$ mod 2. Let $x_1, ..., x_p$, $y_1, ..., y_p$ be

$(2m+1)$-cycles in F which represent sympletic basis of $H_{2m+1}(F:\mathbb{Z})/(\text{Torsion part})$, i.e., basis such that (1) $x_i \cdot y_i = 1$ for all i, (2)$x_i \cdot x_j = 0$, $y_i \cdot y_j = 0$ for all (i,j), and (3)$x_i \cdot y_j = 0$ for $i \neq j$. Note that symplectic basis always exist. Then define $\text{Arf}(K) = \sum_{i=1}^{p} q(x_i)q(y_i) \in \mathbb{Z}_2$.

At the end we review on the definition of the signature of knots. Define *the signature* $\sigma(K)$ of K to be the signature of $A + {}^tA$. Recall that, when k is odd, $\sigma(K)$= $\sigma(F)$.

We now begin the proof of Lemma 2.3.

Proof of Lemma 2.3. The case of $k=0$ is induced from [Kf]. We prove the case of $k \geqq 1$.

We first prove:

Claim. Let F be a Seifert hypersurface for K. We can take $2p$ $(k+1)$-cycles $x_1, ..., x_p$, $y_1, ..., y_p$ in F which represent basis of $H_{k+1}(F:\mathbb{Z})/(\text{Torsion part})$ such that (1) $x_i \cdot y_i = 1$ for all i, (Hence,$y_i \cdot x_i = (-1)^{k+1}$ for all i,) (2)$x_i \cdot x_j = 0$, $y_i \cdot y_j = 0$ for all (i,j), and (3)$x_i \cdot y_j = 0$ for $i \neq j$.

Proof. When k is even, take symplectic basis. When k is odd, we need the following.

Sublemma. *(See e.g. [S].) A symmetric matrix satisfies the conditions that (1)the elements are integers, (2)the determinant is* $\underline{+}1$*, and (3)the signature is zero, then the matrix is congruent to* $\oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Since a matrix which represents the intersection products of $H_{k+1}(F;\mathbb{Z})$ satisfies the condition of the above sublemma, such x_i and y_j exist. The proof of the above Claim is completed.

There exists a Seifert matrix X of K associated with the basis, x_i and y_j, and F. The elements of X are $\theta(x_i, x_j)$ $\theta(x_i, y_j)$, $\theta(y_i, x_j)$, and $\theta(y_i, y_j)$ $(i, j = 1, ..., p)$.

Take an embedding $f : S^{k+1} \times S^{k+1} \hookrightarrow B^{2k+3}$ so that $f(S^{k+1} \times S^{k+1})$ is a tubular neighborhood of the standard $(k+1)$-sphere embedded trivially in B^{2k+3}. We regard S^{k+1} as $D_1^{k+1} \cup D_2^{k+1}$. Let A denote $f(S^{k+1} \times S^{k+1} - \text{Int}\{D_2^{k+1} \times D_2^{k+1}\})$. Let $p_1, ..., p_\mu$ be points in D_2^{k+1}, where μ is a large positive integer. Take a neighborhood $\begin{cases} U_\alpha \\ V_\beta \end{cases}$ of $\begin{cases} D_1^{k+1} \times p_\alpha \\ p_\beta \times D_1^{k+1} \end{cases}$ in B^{2k+3} such that (1)U_i and V_i are diffeomorphic to open $(2k+3)$-balls, and (2)arbitrary two of them don't intersect. Let q be the center of D_1^{k+1}. Let x' (resp.y') denote the homology class which is represented by $q \times S^{k+1}$ (resp.$S^{k+1} \times q$). Note that (1)A cycle representing x' intersects with each U_α at one points and doesn't intersect any V_β, and (2)A cycle representing y' intersects with each V_β at one points and doesn't intersect any U_α.

Take disjoint $(2k+3)$-balls $B_i^{2k+3} (i = 1, ..., p)$ in S^{2k+3}. Take a copy of A in each B_i^{2k+3}, say A_i. Take a copy of U_α (resp.V_β) in B_i, say $U_{i\alpha}$ (resp. $V_{i\beta}$). Take a copy of x' (resp.y') in B_i, say x'_i (resp. y'_i). By using (2k+3)-dimensional 1-handles, take the connected sum of A_i in S^{2k+3}, say A_0. Then ∂A_0 is the trivial $(2k+1)$-knot.

There exists a Seifert matrix X' of the trivial knot associated with the basis, $x'_1, y'_1, ..., x'_p, y'_p$, and A_0. Obviously, X' is $\oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The elements of X' satisfies that (1) $\theta(x'_i, x'_j)$=0 for all (i,j), (2) $\theta(y'_i, y'_j)$=0 for all (i,j), (3) $\theta(x'_i, y'_j)$=$\theta(y'_j, x'_i)$=0 for $i \neq j$, (4) $\theta(x'_i, y'_i)$=1 for all i, and (5)$\theta(y'_i, x'_i)$=0 for all i $(i, j = 1, ..., p)$.

We make a $(2k+1)$-knot $\widetilde{K}$ from the trivial $(2k+1)$-knot by high dimensional pass-moves as in the following paragraphs. Before making $\widetilde{K}$, we prove:

Claim. If a Seifert matrix of $\widetilde{K}$ coincides with that of K, then $\widetilde{K}$ is what we required, i.e., the proof of Lemma 2.3 is completed.

Proof. By the definition of the construction of $\widetilde{K}$, $\widetilde{K}$ is pass-move equivalent to the trivial knot. By Theorem 2.5, $\widetilde{K}$ is cobordant to K.

We take the following pass-move-charts of ∂A_0, carry out the pass-moves, and modify the Seifert matrix X' to X. For the new knots obtained by the following pass-moves, we can take a Seifert hypersurfaces diffeomorphic to A. We can call basis for the new Seifert matrixes x_i and y_j, again.

Step 1. See $\theta(x_i, x_j)$ $(i > j)$. If $\theta(x_i, x_j)$=0, then $\theta(x'_i, x'_j)$=$\theta(x_i, x_j)$. If $\theta(x_i, x_j)$=$\nu \neq 0$, make $|\nu|$ pass-move-charts $\widetilde{U_{ijk}}$ $(k = 1, ..., |\nu|)$ from U_{ik} and U_{jk} so that pass-moves in $\widetilde{U_{ijk}}$ let $\theta(x'_i, x'_j)$=$\theta(x_i, x_j)$.

Here, we have the following. We prove:

Claim. Then $\theta(x'_i, x'_j)$=$\theta(x_i, x_j)$ $(i < j)$.

Proof. Let $i < j$. By Lemma 2.4(1), $\theta(x'_i, x'_j)$= $x'_i \cdot x'_j$+ $(-1)^k$ $\theta(x'_j, x'_i)$ and $\theta(x_i, x_j)$= $x_i \cdot x_j$+ $(-1)^k$ $\theta(x_j, x_i)$. By the definition of x_i and x'_j, $x_i \cdot x_j$=$x'_i \cdot x'_j$(=0). Since $j > i$, $\theta(x'_j, x'_i)$=$\theta(x_j, x_i)$. Therefore $\theta(x'_i, x'_j)$=$\theta(x_i, x_j)$.

Here, note that, by the definition of these pass-moves, each pass-move in $\widetilde{U_{ijk}}$ doesn't change the value of $\theta(*, \dagger)$ except for $\theta(x'_i, x'_j)$ and $\theta(x'_j, x'_i)$ $(i > j)$.

Step 2. See $\theta(y_i, y_j)$ $(i > j)$. If $\theta(y_i, y_j)$=0, then $\theta(y'_i, y'_j)$=$\theta(y_i, y_j)$. If $\theta(y_i, y_j)$=$\nu \neq 0$, make $|\nu|$ pass-move-charts $\widetilde{V_{ijk}}$ $(k = 1, ..., |\nu|)$ from V_{ik} and V_{jk} so that pass-moves in $\widetilde{V_{ijk}}$ let $\theta(y'_i, y'_j)$=$\theta(y_i, y_j)$. Here, $\theta(y'_i, y'_j)$=$\theta(y_i, y_j)$ $(i < j)$ holds by Lemma 2.4(1).

Here, note that, by the definition of these pass-moves, each pass-move in $\widetilde{V_{ijk}}$ doesn't change the value of $\theta(*, \dagger)$ except for $\theta(y'_i, y'_j)$ and $\theta(y'_j, y'_i)$ $(i > j)$.

Step 3. See $\theta(x_i, y_j)$ for any (i, j). If $\theta(x_i, y_j)$=0, then $\theta(x'_i, y'_j)$=$\theta(x_i, y_j)$. If $\theta(x_i, y_j)$=$\nu \neq 0$, make $|\nu|$ pass-move-charts $\widetilde{W_{ijk}}$ $(k = 1, ..., |\nu|)$ from U_{ik} and V_{jk} so that pass-moves in $\widetilde{W_{ijk}}$ let $\theta(x'_i, y'_j)$=$\theta(x_i, y_j)$. Here, $\theta(y'_i, x'_j)$=$\theta(y_i, x_j)$ holds by Lemma 2.4(1).

Here, note that, by the definition of these pass-moves, each pass-move in $\widetilde{W_{ijk}}$ doesn't change the value of $\theta(*, \dagger)$ except for $\theta(x'_i, y'_j)$ and $\theta(y'_j, x'_i)$.

Here, note that, by the definition of these pass-moves, each pass-move in $\widetilde{W'_{ijk}}$ doesn't change the value of $\theta(*, \dagger)$ except for $\theta(y'_i, x'_j)$ and $\theta(y'_j, x'_i)$ $(i > j)$.

Before Step 4 and 5, we prove:

Claim. (1)If k is odd, $\theta(x_i, x_i)$=$\theta(y_i, y_i)$=0. (2)If k is even,we can assume $\theta(x_i, x_i)$, and $\theta(y_i, y_i)$ are even integer.

Proof. (1)By Lemma 2.4(1) and the definitions of x_i, y_j, x'_i and y'_j, $\theta(x'_i, x'_i)$=$\frac{x'_i \cdot x'_i}{2}$=0 and $\theta(y'_i, y'_i)$=$\frac{y'_i \cdot y'_i}{2}$=0. (2) We can change the basis, if necessary, because Arf(K)=0 and the Arf invariant of the trivial knot is zero.

Step 4. See $\theta(x_i, x_i)$. If k is odd, $\theta(x_i, x_i)$=0. Hence $\theta(x'_i, x'_i)$=$\theta(x_i, x_i)$. The case when k is even. If $\theta(x'_i, x'_i)$=0, then $\theta(x'_i, x'_i)$=$\theta(x_i, x_i)$. The case of $\theta(x'_i, x'_i)$≠0. We can put it 2ν.

Make $|\nu|$ pass-move-charts $\widetilde{Z_{ik}}$ ($k = 1, ..., |\nu|$) from U_{ik} and $U_{i(\nu+k)}$ so that pass-moves in $\widetilde{Z_{ik}}$ let $\theta(x'_i, x'_i)$=$\theta(x_i, x_i)$.

Here, note that, by the definition of these pass-moves, each pass-move in $\widetilde{Z_{ik}}$ doesn't change the value of $\theta(*, \dagger)$ except for $\theta(x'_i, x'_i)$.

Step 5.(final step.) See $\theta(y_i, y_i)$. If k is odd, $\theta(y_i, y_i)$=0. Hence $\theta(y'_i, y'_i)$=$\theta(y_i, y_i)$. The case k is even. If $\theta(y_i, y_i)$=0, then $\theta(y'_i, y'_i)$=$\theta(y_i, y_i)$. The case of $\theta(y_i, y_i)\neq$0. We can put it 2ν. Make $|\nu|$ pass-move-charts $\widetilde{Z'_{ik}}$ ($k = 1, ..., |\nu|$) from V_{ik} and $V_{i(\nu+k)}$ so that pass-moves in $\widetilde{Z'_{ik}}$ let $\theta(y'_i, y'_i)$=$\theta(y_i, y_i)$.

Here, note that, by the definition of these pass-moves, each pass-move in $\widetilde{Z'_{ik}}$ doesn't change the value of $\theta(*, \dagger)$ except for $\theta(y'_i, y'_i)$.

We now obtain $\widetilde{K}$. As seen before, we complete the proof of Lemma 2.3. □

Note that, in the above proof, we proved the following Claim 2.2.1.S which is stronger than Claim 2.2.1.

Claim 2.2.1.S. *If (2) of Theorem 2.2 holds, then (1) of Theorem 2.2 holds and there exist pass-move-charts U_i ($i = 1, .., q$) of K_3 such that $U_i \cap U_j = \phi$ ($i \neq j$) and K_1 is obtained from K_3 by the pass-moves in U_i.*

3. A sufficient condition for the realization of pair of odd dimensional knots

In this section we prove:

Proposition 3.1. *Let K_1 and K_2 be $(2k+1)$-knots and X_2 a $(2k+3)$-knot diffeomorphic to the standard $(2k+3)$-sphere ($k \geqq 0$). If we have the condition that*
$$\begin{cases} \text{Arf}(K_1)=\text{Arf}(K_2) & \text{when } k \text{ is even} \\ \sigma(K_1) = \sigma(K_2) & \text{when } k \text{ is odd,} \end{cases}$$
then, for a $(2k+3)$-knot X_1 diffeomorphic to the standard $(2k+3)$-sphere, (K_1, K_2,X_1, X_2) is realizable.

To prove Proposition 3.1, we need some lemmas.

Lemma 3.2. *Let K be an arbitrary n-knot and X_1 and X_2 arbitrary $(n+2)$-knots diffeomorphic to the standard $(n+2)$-sphere ($n \geqq 1$). Then (K, K, X_1, X_2) is realizable.*

Lemma 3.3. *If 4-tuple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is realizable ($n \geqq 1$) and K_2 is cobordant to $\widetilde{K_2}$, then for an $(n+2)$-knot $\widetilde{X_1}$ diffeomorphic to the standard $(n+2)$-sphere, (K_1, $\widetilde{K_2}$, $\widetilde{X_1}$, X_2) is realizable. Furthermore, for an arbitrary Seifert hypersurface F for the n-knots $\widetilde{K_2}$, there exists an immersion $\widetilde{f} : S^{n+2}_1 \coprod S^{n+2}_2 \looparrowright S^{n+4}$ to realize (K_1, $\widetilde{K_2}$, $\widetilde{X_1}$, X_2) and a Seifert hypersurface $\widetilde{V}$ for $\widetilde{X_1}$=$\widetilde{f}(S^{n+2}_1)$ such that $\widetilde{V} \cap \widetilde{f}(S^{n+2}_2)$= $\widetilde{V} \cap X_2$ is the Seifert hypersurface F for $\widetilde{K_2}$.*

Lemma 3.4. *Let $K_1$and K_2 be $(2k+1)$-knots and X_1 and X_2 arbitrary $(2k+3)$-knots diffeomorphic to the standard $(2k+3)$-sphere ($k \geqq 0$). If $K_1$and K_2 are pass-move equivalent and there exist pass-move-charts U_i ($i = 1, .., q$) of K_2 such that $U_i \cap U_j = \phi$ ($i \neq j$) and K_1 is obtained from K_2 by the pass-moves in U_i, then (K_1, K_2, X_1, X_2) is realizable.*

We first prove:

Claim. If Claim 2.2.1.S, Lemma 3.3, and 3.4 hold, Proposition 3.1 holds.

Proof. By Claim 2.2.1.S, (1) there exists a $(2k+1)$-knot K_3 which is pass-move equivalent to K_1 and cobordant to K_2 and (2)there exist pass-move-charts U_i $(i = 1, .., q)$ of K_3 such that $U_i \cap U_j = \phi$ $(i \neq j)$ and K_1 is obtained from K_3 by the pass-moves in U_i. By Lemma 3.4, (K_1, K_3, X_2, X_2) is realizable. By Lemma 3.3, for a $(2k + 3)$-knot X_1, (K_1, K_2, X_1, X_2) is realizable.

Note. Lemma 3.2 is used to prove Lemma 3.3 and 3.4. The latter half of Lemma 3.3 is used in §4. The case when n is even of Theorem 1.3 is induced from Lemma 3.4, obviously. But we prove in §6 Theorem 1.4 stronger than it.

In the rest of this section we prove Lemma 3.2-4.

Proof of Lemma 3.2. Take an embedding $f_a : S_1^{n+2} \coprod S_2^{n+2} \hookrightarrow S^{n+4}$ which defines the trivial $(n+2)$-link. There exists a chart U of S^{n+4} with the following properties (1) and (2).

(1) $\phi : U \cong \mathbb{R}^{n+2} \times \{(u, v)|u, v \in \mathbb{R}\} \cong \mathbb{R}^{n+2} \times \mathbb{R}_u \times \mathbb{R}_v$

(2) $U \cap f_a(S_1^{n+2}) = \mathbb{R}^{n+2} \times \{(u, v)|u = 0, v = 0\}$

$U \cap f_a(S_2^{n+2}) = \mathbb{R}^{n+2} \times \{(u, v)|u = 3, v = 0\}$

We modify the embedding f_a to obtain an immersion $f_b : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ to realize (K, K, T, T), where T is the trivial knot. We put $f_b|S_2^{n+2} = f_a|S_2^{n+2}$. We define $f_b|S_1^{n+2}$ as follows. Take an embedding $g : \Sigma^{2k+1} \hookrightarrow U \cap f_a(S_1^{n+2})$ which defines the n-knot K in S_1^{n+2}. Let F be a Seifert hypersurface for K and a submanifold of $U \cap f_a(S_1^{n+2})$. Let $N_1(F) = F \times [-1, 1]$ be a submanifold embedded in $U \cap f_a(S_1^{n+2})$ such that $F = F \times \{0\}$. We define the subsets E_1, E_2 and E_3 of $N_1(F) \times \mathbb{R}_u \times \mathbb{R}_v = \{(p, t, u, v)|p \in F, -1 \leqq t \leqq 1, u \in \mathbb{R}, v \in \mathbb{R}\}$ as follows.

$$E_1 = \{(p, t, u, v)|p \in F, 0 \leqq u \leqq 1, -1 \leqq t \leqq 1, v = 0\}$$

$$E_2 = \{(p, t, u, v)|p \in F, 1 \leqq u \leqq 2, t = k \cdot \cos\tfrac{\pi(u-1)}{2}, v = k \cdot \sin\tfrac{\pi(u-1)}{2}, -1 \leqq k \leqq 1\}$$

$$E_3 = \{(p, t, u, v)|p \in F, 2 \leqq u \leqq 4, t = 0, -1 \leqq v \leqq 1\}$$

Then the followings hold by the way of the construction.

$$\Sigma = \overline{\{f_a(S_1^{n+2}) - N_1(F)\}} \cup_{\partial N_1(F)} \overline{\{\partial(E_1 \cup E_2 \cup E_3)\} - N_1(F)}$$

is a (n+2)-sphere embedded in S^{n+4}.
Since $\overline{\{\partial(E_1 \cup E_2 \cup E_3)\} - N_1(F)}$ is isotopic to $N_1(F)$ relative to $\partial N_1(F)$, $f_b|S_1^{n+2}$ defines the trivial knot. Both $\Sigma \cap f_b(S_2^{n+2})$ in Σ and $\Sigma \cap f_b(S_2^{n+2})$ in $f_b(S_2^{n+2})$ defines K. We define $f_b|S_1^{n+2}$ so that $f_b(S_1^{n+2})$ coincides with Σ. We obtain f_b to realize (K, K, T, T).

By the following Sublemma 3.5, (K, K, X_1, X_2) is realizable. We prove Sublemma 3.5 and complete the proof of Lemma 3.2.

Sublemma 3.5. Let T be the trivial $(n + 2)$-knot and X_i arbitrary $(n + 2)$-knots $(i = 1, 2)(n \geqq 1)$. If (K_1, K_2, T, T) is realizable, then (K_1, K_2, X_1, X_2) is realizable.

Proof of Sublemma 3.5. Let $f' : S_1^{n+2} \amalg S_2^{n+2} \looparrowright S^{n+4}$ realize (K_1, K_2, T, T). Let $f_i : S_i^{n+2} \hookrightarrow S^{n+4} (i = 1, 2)$ define $(n + 2)$-knots X_i. Let B', B_1 and B_2 be $(n + 4)$-balls in S^{n+4} and $B' \cap B_1 = B' \cap B_2 = B_1 \cap B_2 = \phi$. We can take f' and f_i so that $\text{Im} f_i$ in B_i and $\text{Im} f'$ in B'. Connect $f_i(S_i^{n+2})$ with $f'(S_i^{n+2})$ by $(n+3)$-dimensional 1-handle h_i embedded in S^{n+4} $(i = 1, 2)$, where $h_1 \cap h_2 = \phi$.Take $f : S_1^{n+2} \amalg S_2^{n+2} \looparrowright S^{n+4}$ so that $f(S_i^{n+2})$ coincides with $f_i(S_i^{n+2}) \sharp f'(S_i^{n+2})$. Then f realizes (K_1, K_2, X_1, X_2). □

Proof of Lemma 3.3. Let $f : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ be an immersion to realize (K_1, K_2, X_1, X_2) and V a Seifert hypersurface for $f(S_1^{n+2})$. Let $f(S_2^{n+2})\times D^2{=}X_2\times \{(x,y)|$ $x = r\cdot\cos\theta, y = r\cdot\sin\theta,\ 0 \leqq r \leqq 1, 0 \leqq \theta < 2\pi\}$ be a tubular neighborhood of X_2 in S^{n+4}.

See $V \cap \{f(S_2^{n+2}) \times D^2\}$. It has the following properties.

(1) For any (x,y), $\{\partial\widetilde{V}\} \cap [f(S_2^{n+2}) \times (x,y)]$ in $f(S_2^{n+2}) \times (x,y)$ defines K_2.
(2) For each (x,y), $\widetilde{V} \cap [f(S_2^{n+2}) \times (x,y)]$ is a same Seifert hypersurface G.
(3) For any θ, $\{\partial V\} \cap [f(S_2^{n+2}) \times \{(x,y)|x = r\cdot\cos\theta, y = r\cdot\sin\theta, 0 \leqq r \leqq 1\}]$ is diffeomorphic to $K_2 \times [0,1]$.

Prepare $S^{n+2} \times [0,1]$. Put K_2 and G in $S^{n+2} \times \{1\}$. Put $\widetilde{K_2}$ and F in $S^{n+2} \times \{0\}$. Recall K_2 and $\widetilde{K_2}$ are cobordant. Hence there exists a submanifold P in $S^{n+2} \times [0,1]$ which meets the boundary transversely in P, is diffeomorphic to $K_2{\times}[0,1]$, and meets $S^{n+2} \times \{0\}$ at K_2 and $S^{n+2} \times \{1\}$ at $\widetilde{K_2}$. By an elementary discussion of the obstruction theory, there exists a compact submanifold Q in $S^{n+2} \times [0,1]$ such that $\partial Q{=}F \cup P \cup G$.

We modify V and make the following $\widetilde{V}$. Let $\widetilde{V}$ be a submanifold of S^{n+4} satisfying the following conditions.

(1) $\widetilde{V} \cap [S^{n+4}{-} \{f(S_2^{n+2}) \times D^2\}]$ coincides with $V \cap [S^{n+4}$- $\{f(S_2^{n+2}) \times D^2\}]$.
(2) $\{\partial\widetilde{V}\} \cap [f(S_2^{n+2}) \times (0,0)]$ in $f(S_2^{n+2}) \times (0,0)$ is $\widetilde{K_2}$.
(3) $\widetilde{V} \cap [f(S_2^{n+2}) \times (0,0)] = F$.
(4) For any θ, $\{\partial\widetilde{V}\} \cap [f(S_2^{n+2}) \times (\cos\theta, \sin\theta)]$ in $[f(S_2^{n+2}) \times (\cos\theta, \sin\theta)\ ]$ is K_2.
(5) For any θ, $\{\partial\widetilde{V}\} \cap [f(S_2^{n+2}) \times \{(x,y)|x = r\cdot\cos\theta, y = r\cdot\sin\theta, 0 \leqq r \leqq 1\}]$ is diffeomorphic to $K_2 \times [0,1]$(and $\widetilde{K_2}{\times}[0,1]$).
(6) For any θ, $\{\widetilde{V}\} \cap [f(S_2^{n+2}) \times \{(x,y)|x = r\cdot\cos\theta, y = r\cdot\sin\theta, 0 \leqq r \leqq 1\}]$ defines the above submanifold Q in $[f(S_2^{n+2}) \times \{(x,y)|x = r\cdot\cos\theta, y = r\cdot\sin\theta, 0 \leqq r \leqq 1\}]$.

Note.(i) The above condition (5) holds because K_2 is cobordant to $\widetilde{K_2}$. (ii)$\partial\widetilde{V}$ is diffeomorphic to the standard sphere.

Let $\widetilde{f} : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ be an immersion such that (1) $\widetilde{f}(S_1^{n+2})$ coincides with $\partial\widetilde{V}$, say $\widetilde{X_1}$, and (2) $\widetilde{f}|S_2^{n+2} = f|S_2^{n+2}$.

By the construction of $\widetilde{f}$, the n-knot $\widetilde{f}(S_1^{n+2}) \cap \widetilde{f}(S_2^{n+2})$ in $\widetilde{f}(S_1^{n+2})$ is equivalent to $\widetilde{f}(S_1^{n+2}) \cap [f(S_2^{n+2}) \times \{(x,y)|x = 1, y = 0\}]$ in $\widetilde{f}(S_1^{n+2})$,
to $f(S_1^{n+2}) \cap [f(S_2^{n+2}) \times \{(x,y)|x = 1, y = 0\}]$ in $f(S_1^{n+2})$,
and to $f(S_1^{n+2}) \cap f(S_2^{n+2})$ in $f(S_1^{n+2})$, that is, K_1.

Then we obtain the immersion $\widetilde{f}$ to realize $(K_1, \widetilde{K_2}, \widetilde{X_1}, X_2)$ and the required V.

Proof of Lemma 3.4. Take the pass-move-charts $U_1, ..., U_q$ of K_2. In each pass-move-chart U_i, take

$$D_{1i}^{k+1}= [\tfrac{7}{18}, \tfrac{11}{18}] \times D_x^{k+1}(0) \times D_y^{k+1}(\tfrac{1}{2})$$
$$D_{2i}^{k+1}= [\tfrac{5}{9}, \tfrac{7}{9}] \times D_x^{k+1}(\tfrac{1}{2}) \times D_y^{k+1}(0).$$

Put $S_{1i}^{k+1}= \partial D_{1i}^{k+1}$ and $S_{2i}^{k+1}= \partial D_{2i}^{k+1}$. Note the linking number of S_{1i}^{k+1} and S_{2i}^{k+1} is one. Since $H_{2k+1}(S^{2k+3} - \cup_{i,j}\{S_{ji}^{k+1}\})$=0, a Seifert hypersurface F for K_2 is included in $S^{2k+3} - \cup_{i,j} S_{ji}^{k+1}$ $(i = 1, ..., q, j = 1, 2)$.

Then we have the following.

Claim. *If we attach $(2k+4)$-dimensional $(k+2)$-handles with 0-framing to U_i along S^{k+1}_{1i} and S^{k+1}_{2i} and carry out surgery on U_i (and S^{2k+3}), then (1)U_i changes into the $(2k+3)$-ball again, (2)S^{2k+3} changes into the $(2k+3)$-sphere again, and (3)the new knot in the new $(2k+3)$-sphere is the knot obtained from K_2 by the pass-moves in U_i.*

By the above discussion, the followings hold. In all U_i $(i = 1, ..., q)$, carry out the above surgeries. Then S^{2k+3} changes into the $(2k+3)$-sphere again, and K_2 in the old $(2k+3)$-sphere changes into K_1 in the new $(2k+3)$-sphere. There exists a Seifert hypersurface F for K_2 such that $S^{2k+3} \cap S^{k+1}_{ji}$ for all i, j.

We first construct an immersion to realize (K_2, K_2, T, T) as in the proof of Lemma 3.2. Take U, and Σ as in the proof of Lemma 3.2. Take the pass-move-charts U_i of K_2 in U. Take S^{k+1}_{ji} in $U_i \subset U$. We use the Seifert hypersurface F in the previous paragraph as F in the proof of Lemma 3.2. As we see before, we can take a Seifert hypersurface so that it does not intersect with S^{k+1}_{ji} in U_i for all i, j. We use the Seifert hypersurface as F in the proof of Lemma 3.2. We use these U, Σ and F and construct an immersion $f_b : S^{2k+3}_1 \coprod S^{2k+3}_2 \looparrowright S^{2k+5}$ to realize (K_2, K_2, T, T) as in the proof of Lemma 3.2.

We next construct an immersion $f_c : S^{2k+3}_1 \coprod S^{2k+3}_2 \hookrightarrow S^{2k+5}$ to realize $(K_1, K_2,$T,T). Let h_{1i} be a tubular neighborhood of

$[S^{k+1}_{1i} \times \{(u, v)|u = 0, 0 \leqq v \leqq 1\}] \cup [D^{k+1}_{1i} \times \{(u, v)|u = 0, v = 1\}]$

in $U_i \times \{(u, v)|u = 0, v \geqq 0\}$. Let h_{2i} be a tubular neighborhood of

$[S^{k+1}_{2i} \times \{(u, v)|u = 0, -1 \leqq v \leqq 0\}] \cup [D^{k+1}_{2i} \times \{(u, v)|u = 0, v = -1\}]$

in $U_i \times \{(u, v)|u = 0, v \leqq 0\}$. Make a submanifold Λ from Σ and h_{ji} so that

$\Lambda = \overline{\Sigma - \cup^q_{i=1} (h_{1i} \cap \Sigma) - \cup^q_{i=1} (h_{2i} \cap \Sigma)} \cup^q_{i=1} \overline{\partial h_{1i} - (h_{1i} \cap \Sigma)} \cup^q_{i=1} \overline{\partial h_{2i} - (h_{1i} \cap \Sigma)}$.

Of course, $(h_{ji} \cap \Sigma)$ is $(\{\partial h_{ji}\} \cap \Sigma)$ and is the tubular neighborhood of S^{k+1}_{ji} in U_i.

Then the followings hold by the definition of the construction. (1)Λ is the trivial $(2k+3)$-knot. (2)$\Lambda \cap f_b(S^{2k+3}_2)$ in Λ is K_1. (Because the pass-move is carried out in each pass-move-chart U_i.) (3)$\Lambda \cap f_b(S^{2k+3}_2)$ in $f_b(S^{2k+3}_2)$ is K_2. Here, we take an immersion f_c : $S^{2k+3}_1 \coprod S^{2k+3}_2 \hookrightarrow S^{2k+5}$ so that $f_c(S^{2k+3}_1)$ coincides with Λ and $f_c|S^{2k+3}_2 = f_b|S^{2k+3}_2$. Then f_c is an immersion to realize (K_1, K_2, T, T).

At last, by Sublemma 3.5, (K_1, K_2, X_1, X_2) is realizable. □

4. A necessary condition for the realization of pair of $(4m+1)$-knots

In this section we prove:

Proposition 4.1. *If a 4-tuple of $(4m+1, 4m+3)$-knots (K_1, K_2, X_1, X_2) is realizable $(m \geqq 0)$, then Arf(K_1)=Arf(K_2).*

We first prove the following Lemma 4.2.

Lemma 4.2 *If 4-tuple of $(n, n+2)$-knots $(K^{(1)}_1, K^{(1)}_2, X^{(1)}_1, X^{(1)}_2)$ and $(K^{(2)}_1, K^{(2)}_2, X^{(2)}_1, X^{(2)}_2)$ are realizable $(n \geqq 1)$, then $(K^{(1)}_1 \sharp K^{(2)}_1, K^{(1)}_2 \sharp K^{(2)}_2, X^{(1)}_1 \sharp X^{(2)}_1, X^{(1)}_2 \sharp X^{(2)}_2)$ is realizable.*

Note. Lemma 4.2 includes Sublemma 3.5.

Proof of Lemma 4.2. Take two $(n+4)$- balls B_1^{n+4} and B_2^{n+4} in S^{n+4} so that $B_1^{n+4}\cap B_2^{n+4}=\phi$. Take $g_i : S_1^{n+2} \amalg S_2^{n+2} \looparrowright S^{n+4}$ to realize $(K_1^{(i)}, K_2^{(i)}, X_1^{(i)}, X_2^{(i)})$ so that we take $\text{Im}g_i$ in $B_i^{n+4} (i=1,2)$. Let $X_j^{(i)}$ denote $\text{Im}g_i(S_j^{n+2})$ for conveinience. Connect $X_1^{(1)}$ with $X_1^{(2)}$ by using $(n+3)$-dimensional 1-handle h_{g1} embedded in S^{n+4} to obtain $\widetilde{X_1}= X_1^{(1)}\sharp X_1^{(2)}$. Connect $\widetilde{X_1}\cap\ X_2^{(1)}$ with $\widetilde{X_1}\cap X_2^{(2)}$ by $(n+1)$-dimensional 1-handle h'_{g2} embedded in $\widetilde{X_1}$. Connect $X_2^{(1)}$ with $X_2^{(2)}$ by using $(n+3)$-dimensional 1-handle $h_{g2}= h'_{g2}\times D^2$ embedded in S^{n+4} to obtain $\widetilde{X_2}= X_2^{(2)}\sharp X_2^{(2)}$. Take $g : S_1^{n+2} \amalg S_2^{n+2} \looparrowright S^{n+4}$ so that $g(S_i^{n+2})$ coincides with $\widetilde{X}_i$. □

We prove Proposition 4.1 by the reduction to absurdity. We assume Arf$(K_1)\neq$ Arf(K_2) and induce the absurdity. We may assume that Arf(K_1)=1 and Arf(K_2)=0 without loss of generality.

Note. If $bP_{4m+2}=\mathbb{Z}_2$, it is obvious that the proof is easy. Recall that the subgroup $bP_{4m+2} \subset \Theta_{4m+1}$ is the trivial group for some integers and is $\mathbb{Z}_2$ for some integers (See [KM].) It is known that there exist some integers m such that (i) TS^{2m+1} is not the trivial bundle, i.e., $m\neq$0,1,3, and (ii) bP_{4m+2} is the trivial group (See [Br]).

Let T be the trivial $(4m+1)$-knot. Let K be a $(4m+1)$-knot such that a Seifert hypersurface for K is the plumbing F of two copies of the tangent bundle of the standard $(2m+1)$-sphere and a Seifert matrix associated with F is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (See e.g. P. 162 of [LM] for the plumbing.) Then Arf(K)=1 and for an arbitrary non-vanishing $(2m+1)$-cycle x of F, $\theta(x,x)$ is odd. (See §2 of this paper for the definition of $\theta(\ ,\ )$.)

We prove:

Claim. *The pair of* $(4m+1)$*-knots* (T,K) *is realizable under the hypothesis of the reduction to absurdity.*

Proof. By Proposition 3.1, $(-K_1^*, K)$ and $(T, -K_2^*)$ are realizable. By the hypothesis of the reduction to absurdity (K_1, K_2) is realizable. By Lemma 4.2, $(K_1\sharp\{-K_1^*\}\sharp T, K_2\sharp K\sharp\{-K_2^*\})$ is realizable, i.e., $(K_1\sharp\{-K_1^*\}, K_2\sharp K\sharp\{-K_2^*\})$ is realizable. Since $K_1\sharp\{-K_1^*\}$ is cobordant to the trivial knot and $K_2\sharp K\ \sharp\{-K_2^*\}$ is cobordant to K, (T,K) is realizable by Lemma 3.3. □

Let $f : S_1^{4m+3} \coprod S_2^{4m+3} \looparrowright S^{4m+5}$ be an immersion to realize (T,K). By Lemma 3.3, there exist Seifert hypersurfaces V_i for $f(S_i^{4m+3})$ $(i=1,2)$ such that $f(S_1^{4m+3})\cap V_2$ is the $(4m+2)$-disk D and $V_1\cap f(S_2^{4m+3})$ is the Seifert hypersurface F. Let W denote $V_1\cap V_2$. Then $\partial W = F\cup D$. Then the following holds.We prove:

Claim. *There exists a non-vanishing* $(2m+1)$*-*$\mathbb{Z}_2$*-cycle* x *in* F *which is zero cycle in* W*, i.e.,* x *bounds a* $(2m+2)-\mathbb{Z}_2$*-chain* y *in* W.

Proof. The natural inclusion $F\hookrightarrow\partial W$ induces $H_i(F;\mathbb{Z}_2)\cong H_i(\partial W;\mathbb{Z}_2)$ $(i\neq 4m+2)$. Hence it suffices to prove that Ker $\{H_{2m+1}(\partial W;\mathbb{Z}_2)\to H_{2m+1}(W;\mathbb{Z}_2)\}$ is not the trivial group. Consider the exact sequence: $H_i(\partial W;\mathbb{Z}_2)\to H_i(W;\mathbb{Z}_2)\to H_i\ (W,\partial W;\mathbb{Z}_2)$. We use the following part $(*)$ of the exact sequence: $H_{2m+2}(\partial W;\mathbb{Z}_2)\to H_{2m+2}(W;\mathbb{Z}_2)\to H_{2m+2}\ (W,\partial W;\mathbb{Z}_2)\to H_{2m+1}(\partial W;\mathbb{Z}_2)\to H_{2m+1}(W;\mathbb{Z}_2)\to H_{2m+1}\ (W,\partial W;\mathbb{Z}_2)\to H_{2m}(\partial W;\mathbb{Z}_2)$. We have $H_{2m+2}\ (\partial W;\mathbb{Z}_2)\cong H_{2m}\ (\partial W;\mathbb{Z}_2)\cong 0$ and $H_{2m+1}(\partial W;\mathbb{Z}_2)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$. By Poincaré duality and the universal coeficient theorem, the followings hold. (1)The $\mathbb{Z}_2$-rank of H_{2m+2}

$(W;\mathbb{Z}_2)$ and that of H_{2m+1} $(W,\partial W;\mathbb{Z}_2)$ are same, put it r. (2) The $\mathbb{Z}_2$-rank of $H_{2m+1}(W;\mathbb{Z}_2)$ and that of $H_{2m+2}(W,\partial W;\mathbb{Z}_2)$ are same, put it s. Therefore the sequence $(*)$ become as follows: $0 \to \oplus^r\mathbb{Z}_2 \to \oplus^s\mathbb{Z}_2 \to \mathbb{Z}_2\oplus\mathbb{Z}_2 \to \oplus^s\mathbb{Z}_2 \to \oplus^r\ \mathbb{Z}_2 \to 0$. Therefore the $\mathbb{Z}_2$-rank of $\mathrm{Ker}\{H_{2m+1}(\partial W;\mathbb{Z}_2) \to H_{2m+1}(W;\mathbb{Z}_2)\} = s - r = 1$. □

The $(2m+1)-\mathbb{Z}_2$-cycle x bounds a $(2m+2)-\mathbb{Z}_2$-chain z in S_2^{4m+3}. Let w denote the $(2m+2)$-$\mathbb{Z}_2$-cycle $y \cup_x z$. Let ω denote the homology class of w. We prove:

Claim 4.3. *Consider $\omega \in H_{2m+2}(V_2;\mathbb{Z}_2)$. The $\mathbb{Z}_2$-intersection number $\omega\cdot\omega$ in V_2 is one.*

Proof. Let $W\times\{t|-1 \leqq t \leqq 1\}$ be a tubular neighborhood of W in V_2. The $(2n+1)$-cycle $x\times\{t=1\}$ in S_2^{4m+3} bounds a $(2n+2)$-chain $\widetilde{z}$ in S_2^{4m+3}. Let w_t be the cycle

$(y\times\{t=1\})\cup_{x\times\{t=1\}}\widetilde{z}$.

Let $\partial V_2\times\{s|0\leqq s\leqq 1\}$ be a collar neighborhood of ∂V_2 in V_2. Let w_s be the cycle

$(y-[y\cap\ \{\partial V_2\times\{s|0\leqq s\leqq 1\}\}])\cup_{x\times\{s=1\}}(z\times\{s=1\})$.

Then the followings hold. (1) w_s and w_t are in ∂V_2 and represent the same homology class ω. (2)w_s and w_t intersect transversely at odd points because $\theta(x,x)$ is odd. Therefore $\omega\cdot\omega$ is one. □

On the contrary to the above Claim 4.3, the following Claim 4.4 holds. This is absurdity. We prove Claim 4.4 and complete the proof of Proposition 4.1.

Claim 4.4. *The $\mathbb{Z}_2$-intersection number $\omega\cdot\omega$ is zero.*

Proof. Since V_2 is a codimension one orientable submanifold in the parallelizable manifold $S^{4m+5}-\{\text{one point}\}$ and $\partial V_2 \neq \phi$, V_2 is parallelizable. The fact that $\omega\cdot\omega$ is zero follows from the following elementary Lemma. This lemma is essentially same as in P. 525 of [KM]. In fact, it is proved elementarily without using Sq-operators.

Lemma. *(See e.g. P. 525 of [KM].) Let V be a compact parallelizable $2k$-manifold. For an arbitrary k-homology class $\omega\in H_k(V;\mathbb{Z}_2)$ the intersection number $\omega\cdot\omega$ is zero.*

5. A necessary condition for the realization of pair of $(4m+3)$-knots

In this section we prove:

Proposition 5.1. *If a 4-tuple of $(4m+3,4m+5)$-knots (K_1,K_2,X_1,X_2) is realizable ($m\geqq 0$), then $\sigma(K_1)=\sigma(K_2)$.*

Proof of Proposition 5.1. Let $f: S_1^{4m+5}\coprod S_2^{4m+5} \looparrowright S^{4m+7}$ be an immersion to realize (K_1,K_2,X_1,X_2). We abbreviate $X_i = f(S_i^{4m+5})$ to S_i^{4m+5}. There exist compact oriented $(4m+6)$-manifolds V_1 and V_2 such that $S_i^{4m+5}=\partial V_i\subset V_i\subset S^{4m+7}(i=1,2)$ and V_1 and V_2 intersect transversly. Let W be $V_1\cap V_2$. Then $\partial W=\partial(V_1\cap V_2)=(\partial V_1\cap V_2)\cup(V_1\cap\partial V_2)$ $=(S_1^{4m+3}\cap V_2)\cup(V_1\cap S_2^{4m+3})$. Let $F_1=S_1^{4m+3}\cap V_2$ and $F_2=V_1\cap S_2^{4m+3}$. Then F_i in S_i^{4m+5} is a Seifert hypersurface for K_i $(i=1,2)$. Therefore $\sigma(K_1)-\sigma(K_2)=\sigma(K_1)+\sigma(-K_2)$ $=\sigma(F_1)+\sigma(-F_2)=\sigma(\partial W)=0$. The second equality holds by the definition of the signature. The third equality holds by Novikov aditivity. □

5.A. The proof of Claim 2.2.2.

As we state under Theorem 2.2 in §2, we prove Claim2.2.2 here. We complete the proof of Theorem 2.2.

Since K_2 and K_3 are cobordant, $\begin{cases} \text{Arf}(K_2)=\text{Arf}(K_3) & \text{when } k \text{ is even} \\ \sigma(K_2)=\sigma(K_3) & \text{when } k \text{ is odd.} \end{cases}$ There exist $K'_1 = K_1$, K'_2,..., K'_q, $K'_{q+1} = K_2$ and pass-move-charts U_i of K'_i $(i = 1, ..., q)$ and K'_{i+1} is obtained from K'_i by the high dimensional pass-move in U_i $(i = 1, ..., q)$. If the equality (†) $\begin{cases} \text{Arf}(K'_i)=\text{Arf}(K'_{i+1}) & \text{when } k \text{ is even} \\ \sigma(K'_i)=\sigma(K'_{i+1}) & \text{when } k \text{ is odd} \end{cases}$ holds for $i = 1, ..., q$, then the proof is completed. By Lemma 3.4, the pair of $(2k+1)$-knots (K'_i, K'_{i+1}) is realizable. Therefore, by Proposition 4.1 and 5.1, the equality (†) holds.

6. A necessary and sufficient condition for the realization of 4-tuple of even dimensional knots

In this section we prove Theorem 1.4.

We need the following Lemma 6.1.

Lemma 6.1. *Let T be the trivial $(n+2)$-knot, K'_2 the trivial n-knot, and K_1 a slice n-knot. $(n \geqq 1)$. Then (K_1, K'_2, T, T) is realizable.*

Before the proof of Lemma 6.1, we prove:

Claim. *If Sublemma 3.5, Lemma 4.2 and 6.1 hold, Theorem 1.4 holds.*

Proof. Let K_2 be a slice n-knot, K'_1 the trivial n-knot, X_i an arbitrary $(n+2)$-knot diffeomorphic to the standard $(n+2)$-sphere, and T the trivial $(n+2)$-knot $(i = 1, 2)$. By Lemma 6.1, (K_1, K'_2, T, T) and (K'_1, K_2, T, T) are realizable. By Lemma 4.2, $(K_1 \sharp K'_1, K'_2 \sharp K_2, T \sharp T, T \sharp T) = (K_1, K_2, T, T)$ is realizable. By Sublemma 3.5, (K_1, K_2, X_1, X_2) is realizable.

We prove Lemma 6.1 to complete the proof of Theorem 1.4.

Proof of Lemma 6.1. We define $f : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ by using the k-twist spinning in §6 of [Z]. Prepare $D^{n,n-2}$ in §6 of [Z], and put n there to be $(n+3)$. As written there, regard (S^{n+4}, a $(n+2)$-knot) as $(\partial D^{n+3,n+1} \times D^2) \cup (D^{n+3,n+1} \times \partial D^2)$. Take $D^{n+3,n+1}$ as follows. Recall that (1)$D^{n+3,n+1}$ denote a set of D^{n+3} and D^{n+1} embedded in D^{n+3}, (2) $D^{n+3} \cap D^{n+1} = \partial D^{n+1}$ and ∂D^{n+1} in ∂D^{n+3} is the trivial n-knot. Regard D^{n+3} as $D_s^{n+2} \times [-1, 1]$. Let $D^{n+1} \cap \partial D^{n+3} \subset (D_s^{n+2} \times \{-1\})$. Suppose that $D^{n+1} \cap (D_s^{n+2} \times \{0\})$ in $(D_s^{n+2} \times \{0\})$ defines K_1. Such $D^{n+3,n+1}$ exists because K_1 is slice. Define $f|S_1^{n+2}$ so that $f(S_1^{n+2})$ is the boundary of $D_s^{n+2} \times [0, 1] \times \{\theta_0\}$, where θ_0 is a point in ∂D^2. Define $f|S_2^{n+2}$ so that $f(S_2^{n+2})$ coincides with what is made from D^{n+1} by 1-twist-spinning. Then the following claim holds. we prove:

Claim. The immersion f realizes the 4-tuple of $(n, n+2)$-knots (K_1, K'_2, T, T).

Proof. $f(S_1^{n+2})$ is the boundary of the $(n+3)$-ball $D_s^{n+2} \times [0, 1] \times \{\theta_0\}$. Therefore $f|S_1^{n+2}$ defines the trivial knot T. $f(S_1^{n+2})$ is a 1-twist spun knot. By [Z] 1-twist spun knots are trivial. Therefore $f|S_2^{n+2}$ defines the trivial knot T. By the definition of the construction of f, the n-knot $f(S_1^{n+2}) \cap f(S_2^{n+2})$ in $f(S_1^{n+2})$ is $D^{n+1} \cap (D_s^{n+2} \times \{0\})$ in $(D_s^{n+2}$. Therefore $f(S_1^{n+2}) \cap f(S_2^{n+2})$ in $f(S_1^{n+2})$ defines K_1. The $(n+1)$-disc $D^{n+1} \cap (D_s^{n+2} \times [01])$ is called D_l^{n+1}. By the definition of the construction of f, D_l^{n+1} is in $f(S_2^{n+2})$. By the definition of the construction of f, the n-knot $f(S_1^{n+2}) \cap f(S_2^{n+2})$ in $f(S_2^{n+2})$ is the boundary of D_l^{n+1}. Therefore $f(S_1^{n+2}) \cap f(S_2^{n+2})$ in $f(S_2^{n+2})$ defines the trivial knot K'_2. Therfore f is an immersion to realize (K_1, K'_2, T, T). □

References

[Br] Browder, W, *The Kervaire invariants of framed manifolds and its generalization*, Ann. of Math. **90** (1969), 157-186.

[CO] Cochran,T and Orr,K, *Not all links are concordant to boundary links*, Ann. Math. **138** (1993), 519–554.

[Kf] Kauffman,L, *Formal knot theory*, Princeton University Press Math. Notes **30** (1983).

[Kw] Kawauchi,A edit., *Knot theory (In Japanese)*, Springer Verlag Tokyo (1990 *A survey on knot theory (English translation)*, Birkhäuser (To appear 199∗).

[KM] Kervaire,M and Milnor,J, *Groups of homotopy spheres I*, Ann. Math. **77** (1963), 504-537.

[K] Kervaire,M, *Les noeudes de dimensions supéreures*, Bull.Soc.Math.Fr. **93** (1965), 225-271.

[LM] Lawson, H and Michelson, M, *Spin geometry*, Prinston University Press (1989).

[L1] Levine,J, *Knot cobordism in codimension two*, Comment. Math. Helv. **44** (1969), 229-244.

[L2] Levine,J, *Polynomial invariants of knots of codimension two*, Ann. Math. **84** (1966), 537-554.

[L3] Levine,J, *Invariants of knot cobordism*, Invent.Math. **8** (1969), 98-110.

[O1] Ogasa,E, *On the intersection of spheres in a sphere I*, preprint (Tokyo University).

[O2] Ogasa, E, *The intersection of three spheres in a sphere and a new application of the Sato-Levine invariant*, Proc. of AMS (To appear).

[O3] Ogasa, E, *Some properties of ordinary sense slice links: some answers to the problem (26) of Fox*, Proc.AMS (To appear).

[S] Serre,J.P, *Formes bilinéaires symétriques entiȓes à discriminant ±1,*, Seminaire H. Cartan, 14° **année** (1961/62).

[Z] Zeeman, E, *Twisting spun knots*, Trans. Am. Math. Soc. **115** (1965), 471-495.

Department of Mathematical Sciences, University of Tokyo
Komaba, Tokyo 153, Japan
i33992@ m-unix.cc.u-tokyo.ac.jp

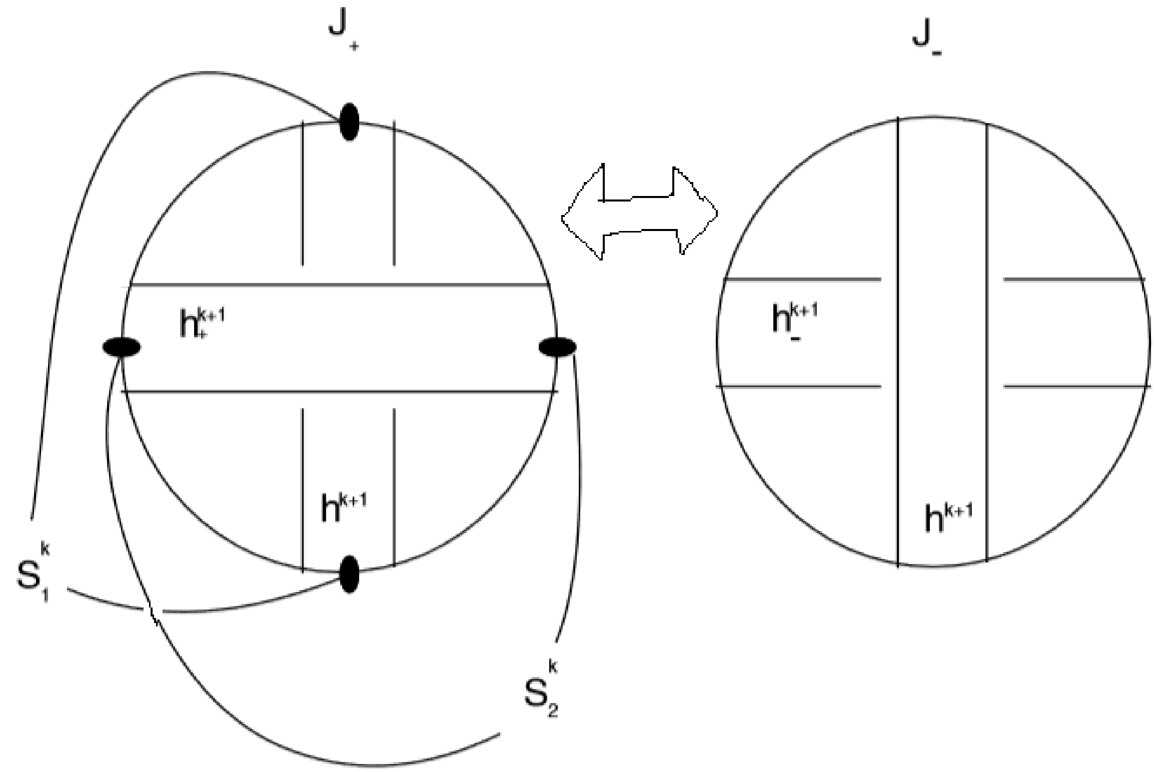

Figure 4.1